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ABSTRACT

This is volume two of a three-volume SMSG junior high school mathematics text. This volume contains the units concerned with geometry. Unit topics include: (1) non-metric geometry; (2) informal geometry; and (3) measurement and approximation. (MP)

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JUNIOR HIGH SCHOOL
MATHEMATICS UNITS

VOLUME II, GEOMETRY

Prepared by the SCHOOL MATHEMATICS STUDY GROUP
Under a grant from the NATIONAL SCIENCE FOUNDATION

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These units were prepared in the summer of 1958 at Yale University by a School Mathematics Study Group writing team. The units were taught in a number of classes during the academic year 1958-59. No major editing of them has been attempted, but typographical and other errors have been corrected.

This volume includes the units concerned with geometry These are units VI, VIII, IX, X in the numbering system originally used.

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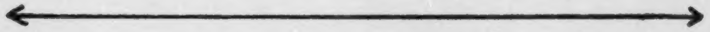
UNIT VI

NON-METRIC GEOMETRY

Sets and the Intersection of Sets

In your study of arithmetic you have learned about properties of number and properties of the operations (addition, subtraction, multiplication, division) on numbers. We are now going to consider the ideas of point, line, and space. A study of these ideas is called geometry.

We shall think of a line as being a representation of certain things, such as the edge of the desk or the edge of a ruler. We use the term line to mean straight line. A line will be represented in a sketch in this way:



The arrows are used here to show that the drawing represents a line extending in both directions without end. We use notation in the next sketch to label a particular point, A, on the line.

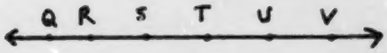
In studying lines we use the ideas of point and set. A set consists of all the things which have some characteristic, or property, in common. For example, the pupils in this class form a set. Each pupil is a member, or element of the set. The even numbers form a set. Three elements of the set of even numbers are 2, 16, and 38. The set of even numbers smaller than 10 are 2, 4, 6, and 8. This set has just four elements. The persons who have been president of the United States are a set. Thomas Jefferson is one element of this set, and President Eisenhower is another element. A set may have no elements, and in that case we call it the "empty set," or the "null set." Probably the set of pupils in your class who are less than eight years old is the empty set.

A line may be thought of as a set of points. Any point on the line is an element of this set.

Two (or more) sets which have elements in common are said to have an intersection. The intersection is also a set. An element is in the intersection set if it is a member of each set. The intersection set of the set of even numbers and the set of whole numbers from 1 to 9 contains the numbers 2, 4, 6, and 8 since these numbers are in both sets. In this room we have a set of boys and a set of people who have a birthday in May. The boys who have a birthday in May are in the intersection set. If no boy in the class has a birthday in May the intersection of the two sets is the empty set.

EXERCISES

1. Write the elements of the set whose members are
   a. The whole numbers greater than 17 and less than 23.
   b. The first names of members of your family.
   c. The cities over 100,000 in population in your state.
   d. The members of your class less than 4 years old.
   e. The names of special kinds of quadrilaterals.
2. Write 3 elements of each of the following sets:
   a. The odd whole numbers.
   b. The whole numbers divisible by 5.
   c. The set of points on the line below, some of which are labeled in the figure:

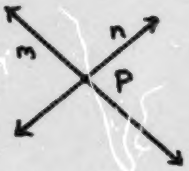


3. Give the elements of the intersections of the following pairs of sets:
   a. The whole numbers 2 through 12 and the whole numbers 9 through 20.

b. The digits in your telephone number and the odd numbers.

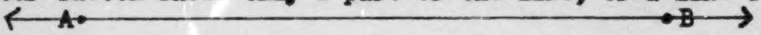
c. The members of the class and the girls with blond hair.

d. Set of points on line m and set of points on line n .



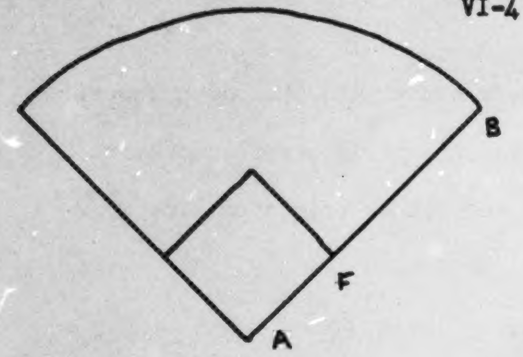
Line Segments and Half-Lines

When we represent a line by a sketch on the blackboard or a piece of paper our sketch shows only a part of the line, or a line segment.



A line segment is determined by 2 points on the line, in this sketch A and B, which are called the endpoints of the segment. The segment, AB, is a set of points. The set consists of A and B and all points between A and B. If the edge of a book is thought of as a line segment, the endpoints are the two corners of the book on that edge. An edge of a foot ruler represents a line segment and the endpoints are usually marked 0 and 12.

On a baseball diamond, we may think of home plate as a point, and first base as a point. Then the part of a line from home plate to first base is a line segment of which the endpoints are homeplate and first base. When this line segment is extended from first base into the outfield, it is called the right-field foul line. A ball which is hit left of it is "fair" and a ball which is hit right of it is "foul." In baseball, one of the foul-lines is a line segment extending from home plate, as one endpoint, through first base into the outfield to the ballpark fence. The point of intersection of the foul-line with the ballpark fence is the other endpoint of this line segment (foul-line). First base, thought of as a point, is a member of the set of points on the line segment (foul-line).



A represents home plate.

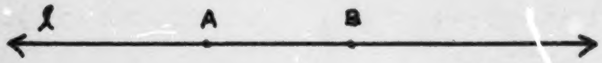
F represents first base.

B represents the intersection of the foul-line and the ballpark fence.

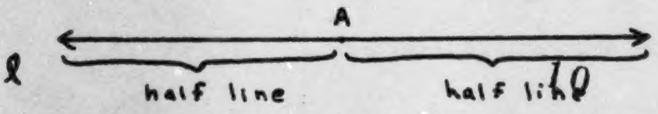
In the figure, AF and AB represent the line segments described above.

In our idea of a line we agree that there is always a third point between any 2 points of the line. This third point is said to be between the 2 points. In the sketch above, F is said to be between A and B. On every line segment there is always a point of the segment between the end-points. Recall that with numbers there is always a third rational number between any two given rational numbers. For example, between $\frac{1}{4}$ and $\frac{1}{2}$ is $\frac{3}{8}$, and also $\frac{7}{16}$. Name two other rational numbers between $\frac{1}{4}$ and $\frac{1}{2}$.

A line segment can always be extended in both directions without limit and this extension of the line segment is called a line, or straight line, as we have seen. The figure shows the line segment AB extended to form a line. A and B and all other points of set AB are elements of the set of points on the line ℓ.



In the figure below, consider the set of points on the line ℓ, and one element of the set, A. A separates the line or set, ℓ, into two half-lines -- the points on the line which are left of A in the figure, and the points on the line which are right of A. We say that A determines 3 sets on the line: the 2 half-lines and the set which contains only the element A. If we join A to one of the half-lines, we call the half-line with A adjoined, a ray.



The intersection of the two rays on a line determined by a point on the the line is the point itself, because the point is on both rays. The intersection set of the two half-lines determined by one point on a line is the empty set. In the example above about the baseball field, the segment from home plate to first base might be thought of as extending without limit in the direction of the outfield. Then the set of points from homeplate through the extended part of the line would be the elements of a half-line. If we join the point represented by home plate to this set, we have a ray.

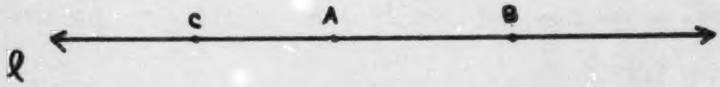
You see that "line" as we use it is merely the way in which we think of many situations in the world about us. The finest machine can never make a line in the mathematical sense, nor can we put the sharpest needle point on a line without its touching many, many points of the line. We say that the sets of points on a line and the line itself are "ideal" ways in which we can describe all kinds of situations in our space.

We shall refer to

(a) the sets of points on line ℓ , or the set ℓ .

(b) the set of points on half-lines or the set of points on rays determined on ℓ by A. To distinguish the two half-lines (or the two rays) we may refer to the half-line from A on B, or the half-line from A on C.

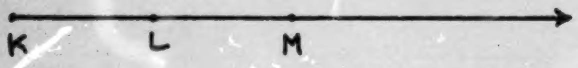
(c) the set of points on the segment AB, or the set AB.



EXERCISES

1. If you think of Main Street in your town as a line segment, what are the endpoints of the line segment? Assume Main Street is straight.
2. Use a ruler to draw a line segment on a piece of paper. Label the endpoints of the segment A and B.
   a. Label as D a point of the set AB. (Make D different from A and B.)
   b. Label as E a point of the set DB. (Make E different from D and B.)
   c. Is E a point of the set AB?
   d. Is E a point of the set AD?
   e. Label F of the set AD. (Make F different from A and D.)
   f. List the segments in your sketch and list also each of the labeled points which are on any segment.
3. a. Extend a segment AB through B so that A is the endpoint of a ray.
   b. Label point C (different from A and B) which is a point of the ray in Part a.
   c. In your sketch is C a point of a ray with endpoint B?
4. On a line label two points A and B.
   a. Name the four half-lines you have sketched
   b. List the different sets of points you have illustrated in the sketch. Can you find 14 different sets?
   c. Label a point C between A and B.
   d. On what half-lines is C?
5. How could you associate the term "ray of light" with the way we are using the term ray?
6. The boundary line between the states of Illinois and Wisconsin may be thought of as a line segment. Think of the segment as extended in an eastward direction. Describe a ray and a half-line associated with this extended segment. (There are two answers.)

7. The figure represents rays with endpoints K, L, and M.



   a. List 3 named points of the ray with endpoint K.

   b. List named points on the half-line determined by K.

   c. Describe two sets of points in the figure of which M is the intersection set.

   d. Describe two sets of points in the figure of which the segment LM is the intersection set.

8. Draw a line, m, and on it mark a point R.

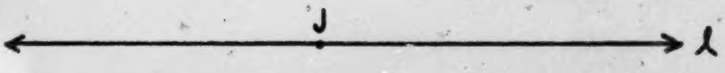
   a. What is the intersection set of the two rays determined by R on m?

   b. What is the intersection set of the two half-lines determined by R on m?

   c. What is the intersection set of one of the half-lines in your sketch with one of the two rays in your sketch? With the other ray?

Lines on a Point

Just as we say "a point lies on a line" we may say "a line lies on a point."



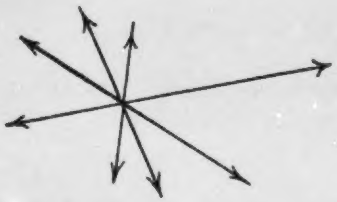
The figure is an illustration of the two sentences:

Point J lies on line ℓ.

Line ℓ lies on point J.

If we think of the line determined by home plate and first base in a ballpark (home plate and first base are thought of as points) we may say that the line lies on home plate or that home plate lies on the line. Also, the line lies on first base, and first base lies on the line.

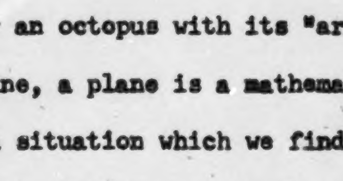
When we considered the set of points on a line, ray, half-line, or segment we thought only in terms of what might be called one dimension in geometry. All of our discussion in this chapter has been about points on a line (in one dimension) except for the ballpark illustration. Now we shall consider the set of lines on a point. You see that there are many lines on a point just as there are many points on a line. Let us first think of the set of lines on a point which are all in the same plane. The figure shows what we mean.

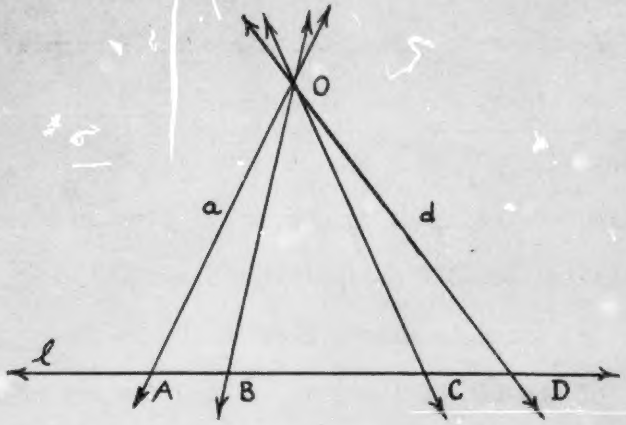


Perhaps this figure makes you think of the spokes of a wheel, a design in a church window, or an octopus with its "arms" all straight.

Like point and line, a plane is a mathematician's way of thinking about the "ideal" of a situation which we find all about us. We may think of a table top, the blackboard, or a soap film on a fine wire frame, as parts of planes. The geometry of the plane is geometry of two dimensions. The points on a plane form a set of points. The lines on a point are a set of lines.

A line separates a plane into two half-planes. In the figure below, point A is a point in one half-plane, and point B is a point in the other half-plane. Point C is on line ℓ, so it is not in either of the half-planes.





In the figure above we have represented a set of lines on a point O, and a set of points on line ℓ. We can see that the intersection of ℓ and a line on O is a point. For example, line a intersects ℓ on A. We can also see that any point on ℓ and point O determine a line on O. For example, D and O determine line d. To a line on O there corresponds a point on ℓ. To a point on ℓ there corresponds a line on O. A relationship like this is called a one-to-one correspondence. Do you see why this is a good term? To a point there corresponds a particular line, and to a line there corresponds a particular point.

EXERCISES

1. Give other examples of a set of lines on a point.
2. Give other examples of a plane (or parts of a plane).
3. Mark point A on a piece of paper. Sketch a set of lines on A.
4. Locate points A and B on your paper.
    a. How many lines may be drawn on point A?
    b. How many lines may be drawn on point B?
    c. Draw four lines on A, and four lines on B.
    d. The lines on A are a set of lines, and you have drawn four

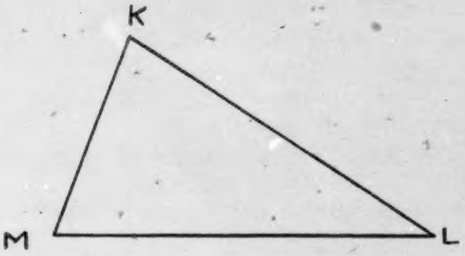
elements of this set. Also, the lines on B are a ___?___ of lines, and you have drawn ___?___ ___?___ of this set.

e. Is some line of the set of lines on A also an element of the set of lines on B? If so, have you drawn it?

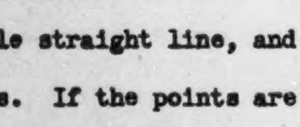
f. A line on A which is also a line on B is the ___?___ ___?___ of the set of lines on A and the set of lines on B.

5. Locate points R, S, and T on a piece of paper, so that they are not on a straight line.
   a. Sketch 4 of the elements of the set of lines on each point.
   b. Include in your sketch the intersection set of each pair of sets in Part a.

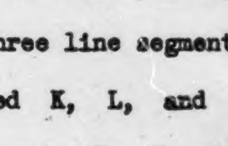
6. A __triangle__ is a figure composed of three points, not all on a single straight line, and the three line segments joining them in pairs. If the points are labeled K, L, and M, the three line segments are labeled KL, LM, and MK. K, L, and M are called the __vertices__ of the triangle. Each is a vertex. Since the points determine the triangle, we can call the triangle, triangle MKL, or more briefly, $\triangle$MKL. (a) Which of the following are triangles?



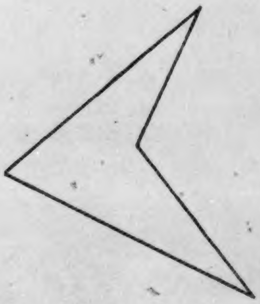
(a)



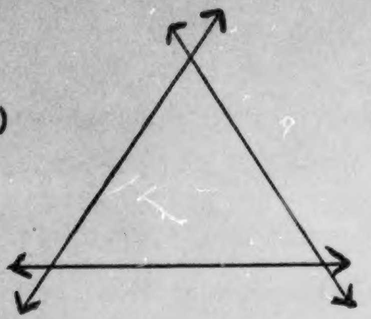
(b)



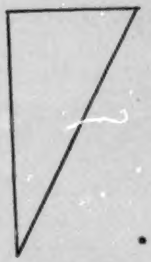
(c)



(d)

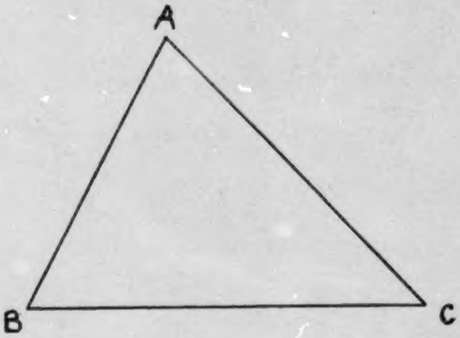


(e)



(b) Which of the figures (a) to (e) contain sets of points which are triangles?

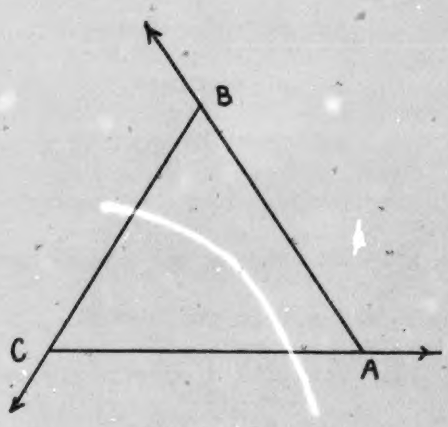
7. a. In the triangle what is the intersection set of AB and AC?



b. Does a triangle contain any rays or half-lines?

c. Draw triangle ABC and extend AB in both directions to obtain a line m. What is the intersection set of AB and m? What is the intersection set of m and the triangle?

8. a. How many half-lines are shown in the figure?



b. How many rays?

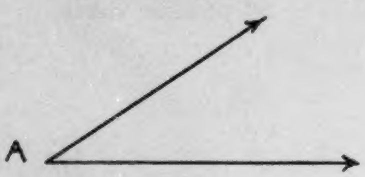
c. On the ray through B of which A is the endpoint there are represented several sets of points as we have discussed them. Name as many as you can.

d. Locate point D, different from A and B, so that it is in as many of the sets of Part c. as possible.

9. Give a definition of a quadrilateral which is similar to the definition of a triangle in Exercise 6.

10. We can think of a triangle as lying on a plane, or determining a plane. Can we conclude that a plane is determined by 3 points?

11. Describe this figure. The set of points on two rays with a common endpoint is called an angle. The figure represents an angle. The endpoint of the rays, A, is the vertex of the angle, and the rays are the sides of the angle.



12. In the figure for Exercise 8, find three angles. Remember the sides must be rays.

13. The opening between the rays which are sides of angle is the inside, or interior of the angle. Draw an angle and call its vertex A. Mark a point B in the interior of the angle. Mark a point C which is not in the interior of the angle.

14. Draw a set of lines on a point P and then draw another line not on the point P which intersects all of the lines of the set. Label points and lines in your sketch so that you can describe the one-to-one correspondence which has been established. (For example, point A might be made to correspond with line a.)

15. Is there a one-to-one correspondence between the pupils and the desks in your room? Why?

16. Describe a one-to-one correspondence between the natural numbers and the even natural numbers. (For example, 1 ↔ 2, 2 ↔ 4, 3 ↔ 6, etc.)

17. Describe a one-to-one correspondence between the points A, B, and C which determine a triangle and the line segments of the triangle. Can you do this in more than one way?

Closed Curves

In newspapers and magazines you often see graphs like these. These graphs are called curves and we shall consider the set of points on a curve. In Figure I, points A, B, C, D, and E are points on the curve, or the curve is on these points. Of course, between any two of these points are many other points. A, B, C, D, and E are elements of the set of points represented by the curve. K, L, and M are points on the curve shown in Figure II, and the curve in II is on the points K, L, and M. A line (or straight line) is a special kind of curve. Since we will call a straight line a special kind of curve, you see the term "curve" is used in mathematics differently from the way it is used in everyday language.

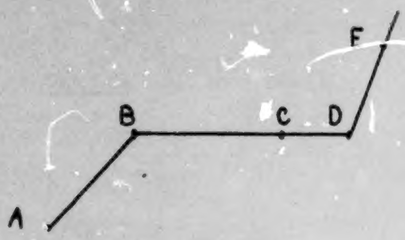


Figure I

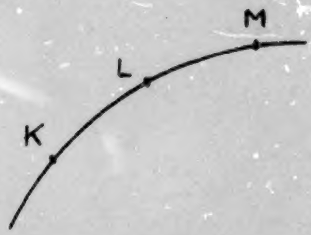


Figure II

Figure III represents the base lines on a ball field. All the points inside the curve made up of the segments AB, BC, CD, and DA are in the infield. If a ball is hit into the infield it is called a "fair" ball. If a ball is hit and then continues it could still be called a fair ball so long as it does not get outside of either of the two rays of which A is the intersection. (What do baseball rules say about this situation?) Since this curve clearly indicates when a ball is fair and when it is foul, it divides the ball field (and this

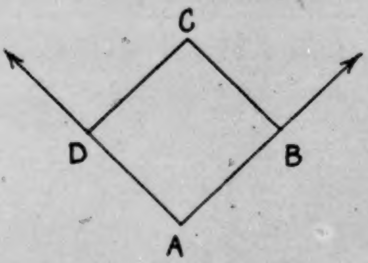


Figure III

extended as far as your imagination chooses) into an inside (fair territory) and an outside (foul territory). This idea of a curve which divides a plane into parts is a very useful one in geometry and some of its applications.

The curves in Figure IV, Figure V, and Figure VI are called simple closed curves.

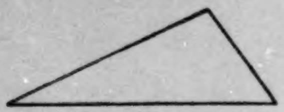


Figure IV

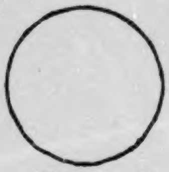


Figure V

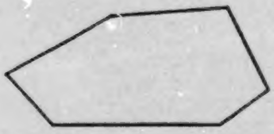


Figure VI

The curves in Figures I and II are not closed curves. Think of all of these curves as plane curves or curves which lie on a plane. A simple closed curve has an outside and an inside and the curve is said to be the boundary of the outside and inside. If point A is inside a simple closed curve and point B is outside, it is not possible to pass in the plane of the curve from A to B without crossing the boundary. A simple closed curve separates a plane into 3 sets, the inside, the outside, and the boundary.

In one dimension a point separates a line into 3 parts or sets. In two dimensions a simple closed curve separates a plane into 3 parts of sets.

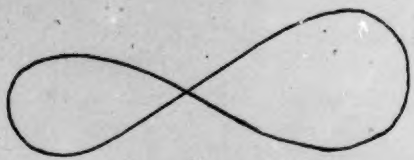


Figure VII

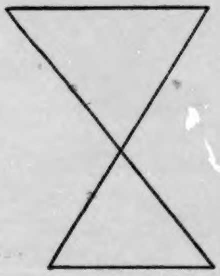


Figure VIII

Figure VII and Figure VIII are closed curves but we do not call them simple closed curves. Each of these curves appears to cross over itself.

In Figure IX, R and S are points in the intersection set of the two curves. R and S are points which lie on both curves. Both curves lie on R and S. T is a point inside the simple closed curve and U is a point outside the simple closed curve. R and S are points on the boundary of the simple closed curve.

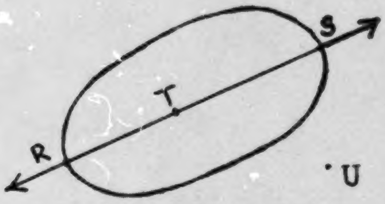
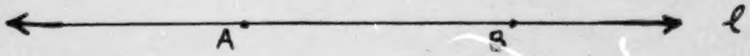


Figure IX

EXERCISES

1.

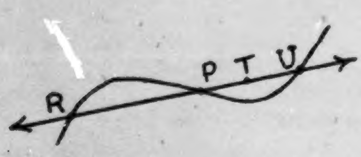


A and B of the set ℓ in the figure, may be thought of as bugs which can crawl only on line ℓ. Describe the simplest set on ℓ which will provide a boundary between A and B, that is, so that A and B cannot touch.

2. Think of A and B as bugs which can crawl anywhere on a plane. Describe a simple set of points in the plane which will provide a boundary between A and B.

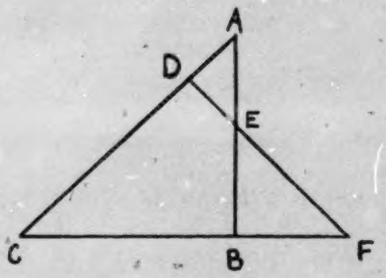
3. Sketch three different simple closed curves which are neither triangles nor circles.

4.

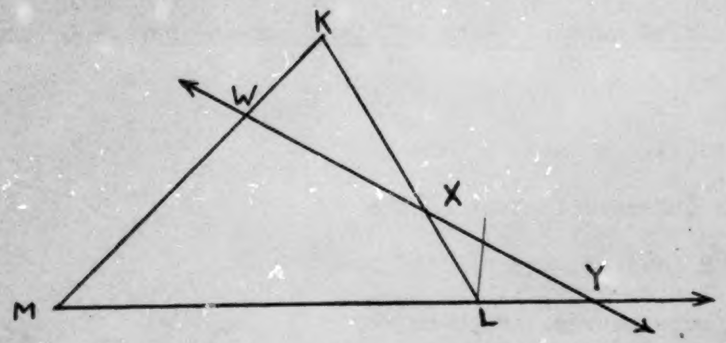


What is the intersection set of the S-curve and the line in the sketch?

5. What is the intersection set of the triangle ABC and the line segment DF?



Exercises 6 through 11 refer to this figure.



6. Describe the positions of W, X, and Y in the figure.

7. Name the triangles in the figure.

8. If possible, locate point I which is not inside any of the triangles in the figure.

9. If possible, locate point H inside two of the triangles in the figure.

10. If possible, locate point G inside triangle MKL but not inside any of the other triangles.

11. If possible, locate point J inside all of the triangles in the figure.

12. If possible, make sketches in which the intersection set of a line and a triangle is

    (a) the empty set   (b) a set of one element

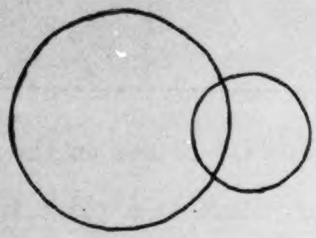
    (c) a set of two elements   (d) a set of three elements

13. If possible, make a sketch in which the intersection set of a line and a circle is

    (a) the empty set   (b) a set of one element

    (c) a set of 2 elements   (d) a set of 3 elements

14. Consider the intersection, in all cases, of a line segment AB and a circle. How many situations must you consider? If A and B are both outside the circle, can the circle and the segment AB have just one point in common?

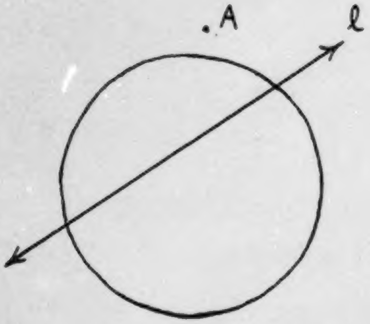
15. 

Sketch two simple closed curves like these for each part of this exercise.

a. Shade the intersection of the sets inside both curves.

b. Shade the intersection of the sets outside both curves.

c. Shade the intersection of the sets outside one curve and inside the other.

16. 

Draw a figure like the one shown.

a. Shade the intersection set of the set of points inside the circle and the set of points on the half-plane which contains A.

b. Shade the intersection set of the set of points outside the circle and the set of points in the half-plane which does not contain A.

c. Describe in your own words the set of points which are not shaded in either Part a. or Part b.

17. Sketch a circle and a quadrilateral for which the intersection set consists of

(a) 4 points (b) 6 points

(c) the largest number of points possible.

18. A line separates a plane into 3 parts or sets. What names would you suggest for these parts?

19. How would you define the inside and the outside of an angle? See the definition of angle, Exercise 11, page 12.

Planes on a Line

If you draw a line on the blackboard you have a representation of the mathematician's idea of a line lying on a plane. All points on the line lie on the plane. A line separates the set of points in a plane into 3 sets: the points on each of the two half-planes and the points on the line. If A is in one half-plane and B is in the other, as in Figure 1, the intersection set of AB and the line is a point. What can you say if A and B are in the same half-plane? See Figure 2.

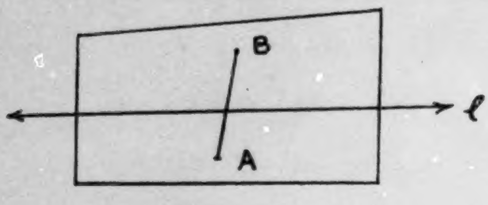


Figure 1

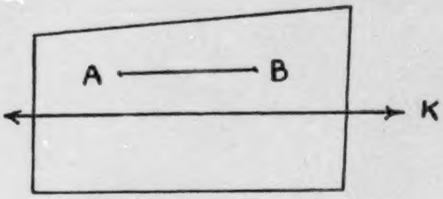


Figure 2

A simple closed curve also separates the plane on which it lies into 3 sets, as we have seen.

If a line lies on a plane, we can also say that the plane lies on the line. If A and B are points on a plane, then the plane lies on the line through A and B.

There is a set of planes on a line. This idea is represented in Figure 3. If you fold a piece of paper and think of the crease as a line, you have a model of two half-planes on a line. Could you make a model of two planes on a line? Do you notice that in dealing with two planes our attention has been called to geometry in three dimensions?

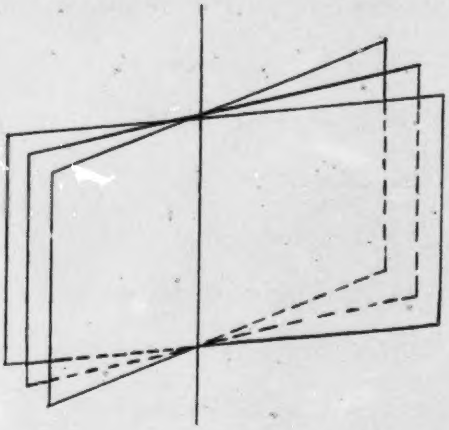


Figure 3

Figure 4 shows two planes on a line k and a second line m which intersects each of the planes in a point. If two planes have an intersection set, then the intersection set is a line. The intersection set of a plane and a line, which does not lie on the plane, is a point, unless the intersection set is the empty set.

Lines like k and m in Figure 4, which have different directions and of which the intersection set is a null set, are called skew lines. Skew lines do not lie on the same plane. Do you see in the room a pair of line segments which if extended would be skew lines?

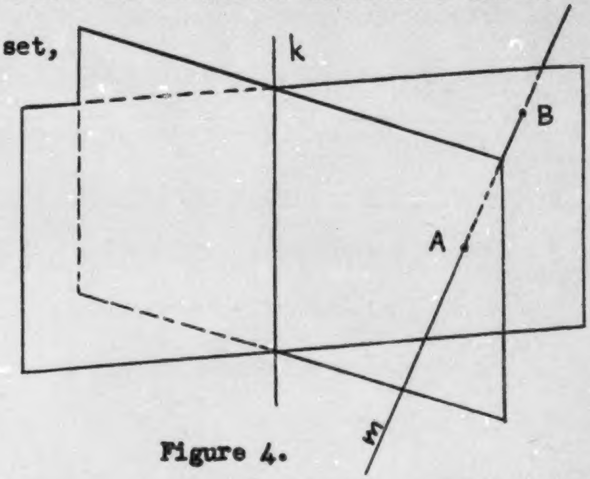


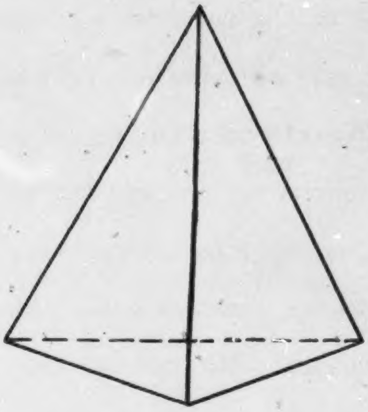
Figure 4.

EXERCISES

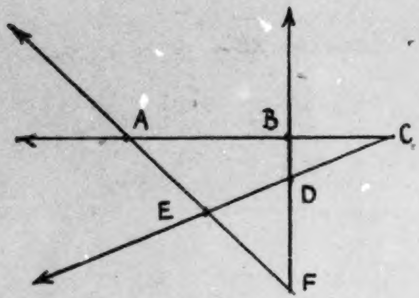
1. Think of the front wall of the classroom, the ceiling, and the side wall as representing 3 planes.
    a. Describe the intersection set of the planes represented by the ceiling and the front wall.
    b. Describe the intersection set of the planes represented by the front wall and the side wall.
    c. Describe the intersection set of all 3 planes.
2. Think of a case where points A and B are outside a plane and on opposite sides of a plane.
    a. Describe the intersection set of line segment AB and the plane.
    b. If A and B are outside a plane and on the same side of a plane, what is the intersection set of line segment AB and the plane?

3. Which pairs of lines determined by the intersection sets of the ceiling, the floor, and 4 side walls appear to be skew lines?

4. Describe several examples of a set of planes on a line.
5. Describe several examples of two skew lines.
6. What is the intersection set of two skew lines?
7. A plane separates the set of points in space into 3 parts. What are they? Suggest names for the parts.
8. Make a model of 3 planes which lie on the same line.
9. Describe the faces (sides, top, and bottom) of a chalk box and the edges in terms of sets and intersection sets. How many of the intersection sets are lines? How many of the intersection sets are points?
10. A chalk box is a solid with 6 faces. The figure shows a tetrahedron, which is a solid of 4 faces. Describe the faces and edges of a tetrahedron in terms of sets and intersection sets. How many of the intersection sets are lines? How many are points?



Special Figures



In the figure we have 4 lines and 6 marked points. On each line there are 3 of the points and on each point there are 2 of the lines. In projective geometry this figure is called a complete quadrilateral. Let us see how this figure can be thought of as a diagram for placing 6 people (the points) on 4 committees (the lines) so that all persons are treated exactly alike. Each person is on the same number of committees and no person is on the same two committees as any other person. Let us consider A and C for example.

A and C are each on 2 committees (and only 2).

A is on one and only one committee with B, with C, with E, and with F.

C is on one and only one committee with B, with A, with E, and with D.

On each of A's committees there are 2 other members shown.

On each of C's committees there are 2 other members shown.

An even more "democratic" committee diagram can be shown for 10 people and 10 committees. With this diagram we can place each person on 3 and only 3 committees and on each committee there will be 3 and only 3 of the 10 people. We might say that this represents real democracy in committee structure. The diagram of 10 points (people) and 10 lines (committees) is called the Desargues configuration. This is a very remarkable figure and is important in the study of projective geometry. Projective geometry arose from the interest of artists in perspective drawing. One of these artists was Leonardo da Vinci.

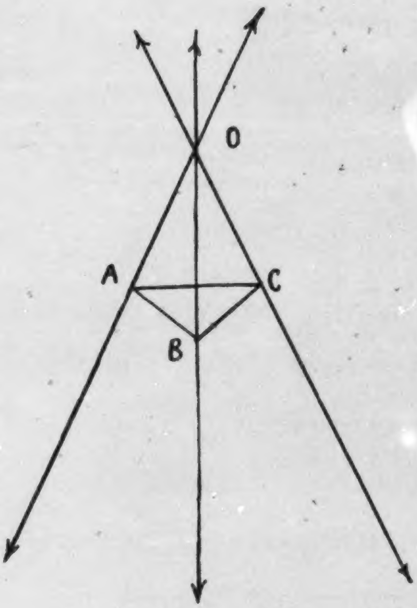
The exercises which follow are concerned with the Desargues configuration.

EXERCISES

1. In the figure there are three lines on a point O and on each of these lines is one of the vertices of the triangle ABC.

   a. Copy this figure on a separate piece of paper.

   b. Draw a second triangle A'B'C' so that A' is on OA extended, B' is on OB extended, and C' is on OC extended. (If possible choose the positions of A'B'C' so that AB extended intersects A'B' extended on your paper.)



   c. Label the points of intersection of the lines as follows:

      line on AB meets line on A'B' in D

      line on AC meets line on A'C' in E

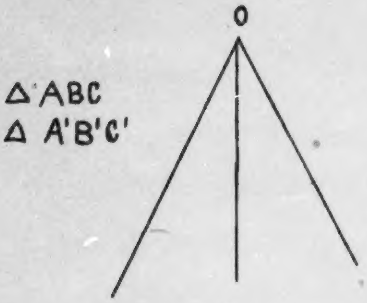
      line on BC meets line on B'C' in F.

   d. Do the points D, E, and F appear to lie on the same straight line? If not, try your drawing again. Draw the line on D, E, and F.

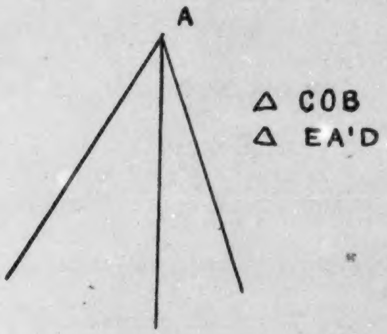
2. Make several drawings like that in Exercise 1.

3. In the completed figure in Exercise 1, you will notice that you have drawn 10 lines (or parts of them) and you have labeled 10 points. Examine your sketch to see if on each of the 10 lines lie 3 of the labeled points and on each of the 10 points lie 3 of the 10 lines you have drawn.

4. One of the remarkable aspects of the figure is that each point and each line play exactly the same role. For example, we may think of point A as the "beginning" point on which 3 lines lie (we used O for this point in Exercise 1). Then we can find on the 3 lines through A the vertices of the triangles COB and EA'D. A listing may help you see this.



lines on segments of
△s meet in D, E, F



lines on segments of
△s meet in C', F, B'

Shade the △ s COB and EA'D in your Desargues figure. It is not necessary to mark any new point nor to extend any line to observe this.

5. Follow through the steps in Exercise 4, starting with point D. What are the vertices of the 2 triangles? In what points do the lines on the segments of the triangles intersect? Do these points lie on a line in your figure?

UNIT VIII

INFORMAL GEOMETRY I

Look around your classroom. The four walls, floor, and ceiling are parts of different planes. The planes intersect in lines, and some of the lines intersect in points. Perhaps your classroom has more than four walls, and perhaps the ceiling is made up of parts of several planes. The blackboard is probably in a different plane from the plane of the wall.

1. Find surfaces in the room and its furniture which are parts of different planes.
2. Find lines which are intersections of planes, and points which are intersections of lines.
3. Can you find three planes which intersect in the same line? Could you, if your room had a revolving door?
4. What planes can you find which do not intersect? What lines can you find that do not intersect?
5. Can you find some surfaces which are curved, not plane?
6. Look out of the window and see how many parts of planes you can find in the roofs of houses and other buildings. How many different planes enclose the whole house? Are any of the surfaces of the house curved, rather than plane?

In geometry we study the properties of space, the relationships between points, lines, planes, curves, surfaces, and solids, and the sizes and shapes of objects. Anyone can plainly see that we cannot develop space travel without knowing about space. Can _you_ see it? Few people realize that we also use geometry to work out the best way for the Army to order shirts.

Imagine a navigator on a Coast Guard vessel which receives an S.O.S. from a sinking ship. From the information he receives from the radio operator he must be able to mark on his chart the position of the ship, and tell his captain in which direction he should steer. They must also know how fast they must travel in order to cover a certain distance before the ship sinks.

Some of the things you will study in geometry were already discovered by the Egyptians and Babylonians almost 4000 years ago. The Greeks, from about 600 B.C. to 200 B.C., made very important advances in geometry. The book of Euclid (about 300 B.C.) is still the basis of the geometry you will study in school. Archimedes, the greatest mathematician of antiquity, and others, such as Pythagoras, Eudoxus, Eratosthenes, and Apollonius, made very important contributions to geometry. After a long period during which very little new knowledge was added, geometry was revolutionized in the 17th century by Pascal, Descartes, and Fermat in France, who discovered how to apply algebra to geometry. The 19th century was a golden age of geometry in which completely new concepts of space were developed by Gauss and Riemann in Germany, Bolyai in Hungary, and Lobachevsky in Russia. About the beginning of the 20th century Poincare and Lebesgue in France and Hilbert in Germany introduced new ways of looking at the very fundamentals of geometry. Important discoveries are being made today by mathematicians in many parts of the world. We may mention, among others, Bing, Moise, Busemann, Milnor, and Nash in the United States, Pontryagin and the two Alexandrovs in Russia, Whitehead in England, Cartan, Leray, Serre, and Thom in France, Hopf and de Rham in Switzerland, Chern from China (now in U.S.), Yamabe from Japan (now in U.S.), and Papakyriakopoulos from Greece (now in U.S.).

There are still many important unsolved problems in geometry. For example, suppose you have a lot of marbles all exactly the same size. How should you pack them in a large container in the best possible way, that is, so as to get as many marbles as you can into the given volume? Nobody knows for sure. We know a very good way to pack the marbles, but no one has ever proved that it is the best possible.

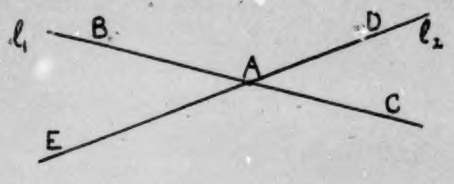
7. Find out where the word "geometry" comes from.
8. Look up one of the mathematicians mentioned above in an encyclopedia or *Bell, Men of Mathematics*, and write a short report on his life and at least one of his important discoveries.

Part A. SETS OF THREE LINES

When a mathematician begins an investigation, he usually starts with a very simple case. After he feels that he understands this case, he may then proceed to more complicated situations. In order to get a feeling for spatial relationships, let us begin by studying figures formed by three lines. In these two chapters, when we speak of a line, we always mean a complete straight line, extended as far as we please in both directions.

When we discuss geometrical objects such as points, lines, and angles, we find it convenient to give them names. Of course, we could call them "Joe" or "Wilhelmina", but we usually use just letters from the alphabet, and speak of the lines m and r, or the points A, P, and Y. Sometimes we use the same letter and attach different *subscripts*, small numerals written below and to the right, such as "ℓ_1", "ℓ_2", etc. The numerals here are simply labels to distinguish different lines or points.

1. Draw two lines, and call them ℓ_1 and ℓ_2. Do ℓ_1 and ℓ_2 intersect in your drawing? If they do not, draw them again so they do intersect. Call their point of intersection point A. (Make the lines long enough to make it easy to measure the angles.)



2. How many angles are formed by ℓ_1 and ℓ_2?
3. Call the rays on ℓ_1 AB → and AC →. (Recall that AB → means the ray with A as endpoint and containing point B.) Call the rays on ℓ_2 AD → and AE →. Some of the pairs of angles which are formed have a common side. These are called adjacent angles. The other two pairs are called vertical angles.

   Each line divides the plane into two half-planes. One of each pair of vertical angles is the common part of two of these four half-planes, and the other vertical angle is the common part of the other two half-planes.
4. Measure the two angles of each pair of vertical angles with your protractor, and compare the measures. What is the relation between these measures?
5. Compare your results with those obtained by your classmates. What relation between the measures of vertical angles is true for each pair?

You have found by experiment, a certain relation to be true in a large number of cases. Let us see why this relation must be true in all cases.

6. a. What is the sum of the measures of angles DAB and BAE?

   b. What is the sum of the measures of the angles CAE and BAE?

c. If the measure of angle DAB is 130 degrees, then what is the measure of angle BAE? What is the measure of angle CAE?

d. Let x, y, and z be the measures of angles DAB, BAE, and CAE, respectively.

State your answers to parts (a) and (b) in the form of equations:

$x + y =$, $z + y =$

If y is known, how can you calculate x and z? Express your answer in the form of equations.

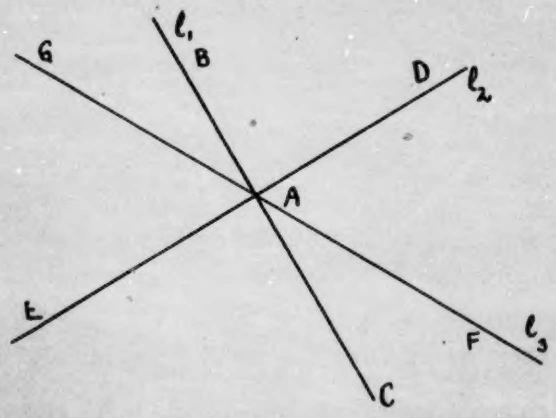
e. Does the relation which you found between x and z in part (d) agree with the result you obtained experimentally in exercise 4 above?

State the result in the form of a general principle by completing the following sentence:

Principle 1. When two lines intersect, angles in each pair of vertical angles formed ________________________.

Compare your statement with those of your classmates. After discussion in class, try to write a clear and precise statement of the principle which all of you find satisfactory.

7. Draw another figure like the one you drew for exercise 1, and name the rays in the same way. Then draw another line on A and call it ℓ_3. Call the other ray, on ℓ_3 between $\overrightarrow{AD}$, and $\overrightarrow{AC}$, $\overrightarrow{AF}$. Call the other ray between $\overrightarrow{AB}$ and $\overrightarrow{AE}$, $\overrightarrow{AG}$. These three lines separate the plane into six regions.



In Exercises 8-12, consider the angles which are the boundaries of these six regions.

8. Start with angle BAD and name successive adjacent angles in clockwise order. How many of these angles are there altogether?

9. How many pairs of vertical angles can you find? Name each pair of vertical angles.

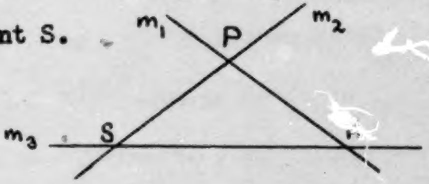
10. What is the sum of the measures of six angles formed by the rays on ℓ_1, ℓ_2 and ℓ_3?

11. How many angles lie on one side of ℓ_1? What is the sum of their measures?

12. Are the answers for exercise 11 true for ℓ_2? For ℓ_3?

13. Draw two lines, m_1 and m_2, which intersect. Call the point of intersection point P.

14. Draw a third line, m_3, which intersects m_1 and m_2, but not in the point P. Call the intersection of m_1 and m_3 point R, and the intersection of m_2 and m_3 point S.

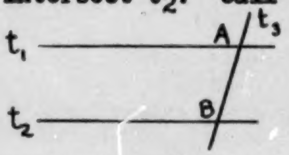


Ex 14 - 17

15. What is the name of the set of points made up of segments SP, PR, and RS?

16. How many angles in the figure have S as vertex? R? P? What kinds of angle pairs can you find?

17. Draw two lines t_1 and t_2 so they do not intersect. Draw a third line t_3 which intersects t_1 and call the point of intersection point A. Does t_3 intersect t_2? Call their point of intersection point B.



18. How many pairs of vertical angles are there in your figure? How many pairs of adjacent angles?

19. In what way are the last three figures you have drawn alike? In what way are they different? Can you think of a fourth way to draw a set of three lines? If so, draw three lines in this way.

20. The set of intersections by pairs of three lines in a plane may be _?_ points, _?_ points, _?_ points, or _?_ points.

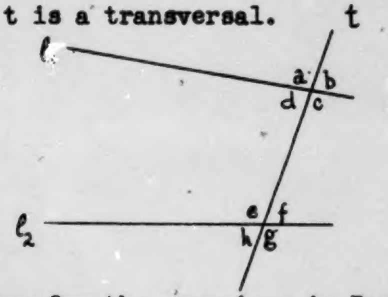
21. Look around you and find illustrations of sets of three lines like those in the figures you drew.

22. Imagine two lines in space. Is there any possible relation which could not occur in a plane?

23. What figures can be formed by the intersections of 3 lines in space?

Part B. TWO LINES AND A TRANSVERSAL

When a line intersects two other lines in distinct points, it is called a transversal of those lines. In the figure, t intersects lines ℓ_1 and ℓ_2, so line t is a transversal.



Ex. 3 - 8

1. In the figures you drew for the exercises in Part A, which ones show transversals?

2. In each figure in Part A which has a transversal, name the transversal and tell what lines it intersects.

3. In the figure above, how many angles are formed? How many pairs of vertical angles are there? What do you know about their measures? (You can say: "$\angle a \overset{m}{=} \angle c$". This means the measures of the angles are the same.)

4. What pairs of adjacent angles are there in the figure?

5. For what pairs of angles is the sum of their measures 180 degrees?

We give special names to some other kinds of angle-pairs in a figure like this. One kind is illustrated by angles b and f.

Notice that one of the rays which forms angle b is a part of a ray which forms angle f, and both angles are on the same side of the transversal t. Angles placed in this way are called corresponding angles.

6. What pairs of corresponding angles can you find in the above figure? There are four pairs in all.

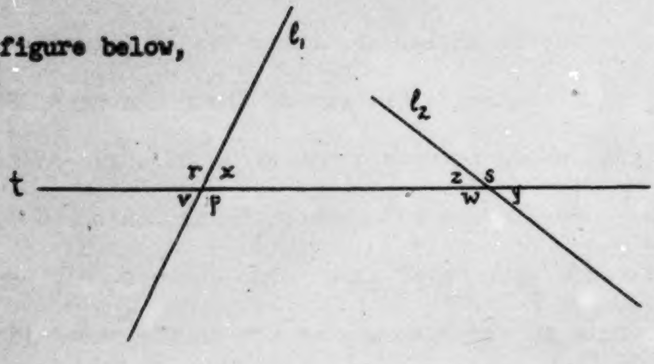
How can we distinguish the angles a, b, g, and h from angles c, d, e, and f? Angles a and b are on the opposite side of ℓ_1 from the intersection of ℓ_2 and t. They are called exterior angles. Similarly, angles g and h are exterior angles. Angles c and d are on the same side of ℓ_1 as the intersection of ℓ_2 and t. These angles and e and f are called interior angles. Angles d and f are interior angles with different vertices on opposite sides of the transversal t. They are called alternate interior angles.

7. Name the pairs of alternate interior angles in the figure above. There are only two pairs.

8. Name four pairs of angles in the above figure which have equal measures, no matter how the figure is drawn. State the reason why they are equal in measure.

Two angles whose measures add up to 180 degrees are called supplementary.

9. In the figure below,



name an angle which forms with angle x a pair of

a. vertical angles

b. corresponding angles

c. alternate interior angles

d. supplementary angles

10. In this figure name:

a. four pairs of vertical angles

b. four pairs of corresponding angles

c. two pairs of alternate interior angles

d. eight pairs of supplementary angles

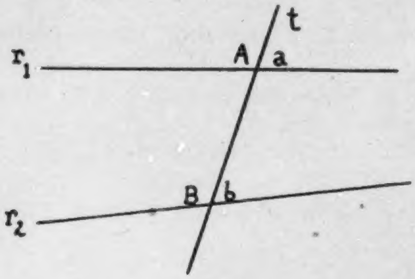
Notice how carefully we defined corresponding and alternate interior angles. Remember that we are discussing ideal points and lines, and that the streaks of printers ink are only meant to suggest a mental picture which you do not actually see. That is why we try to state the relations in terms which do not refer to the drawing. In mathematics we must say exactly what we mean, and must make our definitions as clear and precise as we can.

Part C. PARALLEL LINES AND CORRESPONDING ANGLES

Imagine that you are a civil engineer who must build roads intersecting a highway. In the figure below the highway is represented by the transversal t and the roads are represented by the lines r_1 and r_2.

You want to build the roads so that they don't intersect. Two lines in the same plane which do not intersect are called parallel. Railroad tracks illustrate parallel lines.

You wish to tell your workers what angles the roads should make with the highway. Make a drawing with points A and B about $1\frac{1}{2}$ inches apart.



1. Make the measure of angle a 70 degrees and that of angle b 40 degrees. Do the roads intersect? If so, on which side of t?
2. Draw another figure with the measure of angle a 30 degrees and that of angle b still 40 degrees. What happens now?
3. Make at least ten experiments of this kind with various measures for the angles a and b. Record your results like this:

Measure of Angle a in degrees	Measure of Angle b in degrees	Intersection of r_1 and r_2
70	40	right of t
30	40	left
40	40	?

   Can you find a general law which fits your observations? Compare your results with those of your classmates.

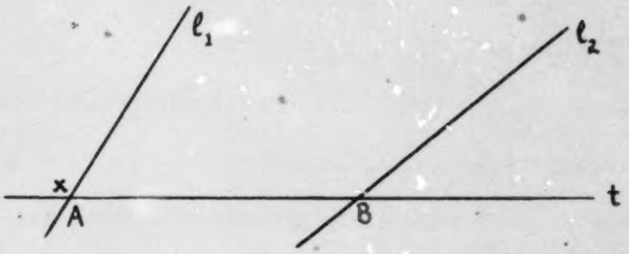
4. What kind of angle pair have you been measuring?
5. Make some more experiments to test the general law which you discovered in exercise 3. In each case first choose the measures of the angles, then predict whether r_1 and r_2 will intersect and if so, where, and finally make a drawing to check your prediction.

6. State your result by filling in the following sentence:

   Principle 2. When a transversal intersects two lines in the same plane, and ______________________________, then the lines are parallel.

   Discuss your statement with your fellow students. Try to arrive at a statement which is clear and precise and satisfies the class.

7. In the figure below, what angle forms with angle x a pair of corresponding angles? Label it angle p. If angles x and p have the following measures, do ℓ_1 and ℓ_2 intersect above or below t or are they parallel?



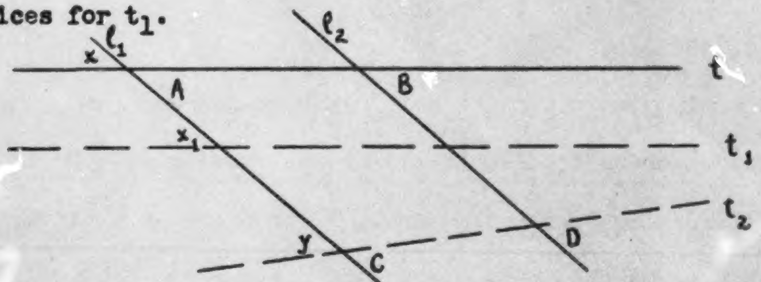
	Measure of Angle x in degrees	Measure of Angle p in degrees	Intersection of ℓ_1 and ℓ_2
a.	120	140	
b.	120	120	
c.	120	90	
d.	90	120	
e.	90	90	
f.	90	70	
g.	70	90	
h.	70	70	
i.	70	30	
j.	30	30	
k.	30	70	
l.	30	10	

Check your predictions by drawing the figures.

State a general principle for deciding, on the basis of the measures of a pair of corresponding angles, whether the lines will meet on the same side of the transversal as this pair of angles or on the other side.

> **Principle 3.** When a transversal intersects two lines in the same plane and, of a pair of corresponding angles, the exterior angle has the larger measure, then
>
> ____.
>
> If, however, the interior angle of the pair has the larger measure, then ____________________________.

8. Draw the figure below so that angles x and p are 40 degrees in measure. Measure the distance from A to B. Now draw another transversal t_1 to ℓ_1 and ℓ_2 so that the measure of the angle x_1 formed by t_1 and ℓ_1 is 40 degrees. What is the distance between the intersections of t_1 with ℓ_1 and ℓ_2? Try the experiment with several choices for t_1.



9. Using the same figure draw transversals t_2 with various values for the measure of angle y. Make your experiments systematically, making the measure of angle y successively, 10, 20, 30, 40, 50, ..., 170 degrees and measure the distance from C to D each time. Record your results in a table.

Measure of Angle y in degrees	Distance from C to D in inches
10	
20	
30	
.	
.	
.	
170	

Try various positions for the point C for each choice of the size of angle y. Does the distance from C to D depend on the position of C or on the angle y? For which value of the measure of angle y is the distance from C to D the least?

10. Try the same experiment with the distance from A to B twice as large as in the first case. Compare your two tables for the distance from C to D.

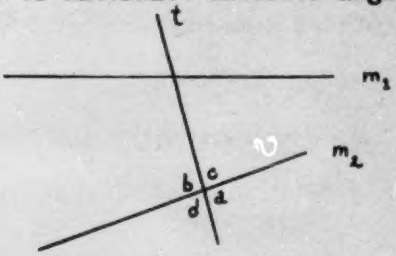
11. Draw a line parallel to a given line so that the minimum distance between the intersections of any transversal with the two lines is one inch.

12. Find in your classroom some parallel lines. Lay a yardstick or a ruler across them so as to form a transversal. Do it so that the intersections with the parallel lines are at a minimum distance apart. Measure this distance in each case.

Part D. PARALLEL LINES AND ALTERNATE INTERIOR ANGLES

You have just found a method for deciding whether two lines intersect by drawing a transversal to the lines and measuring a pair of corresponding angles. Let us see whether you can discover a criterion using alternate interior angles. We do this by applying a favorite device of the mathematician, namely by reducing the problem to the previous case.

1. In this figure, the transversal t intersects the lines m_1 and m_2. Which angle forms with b a pair of alternate interior angles? Label it angle e.



2. Suppose that the measure of angle b is 30 degrees. What are the measures of angles c, d, and a? Which of the previous three principles applies here?
3. Which of these forms with angle e a pair of corresponding angles?
4. If the measure of angle e is 20 degrees, will the lines m_1 and m_2 intersect, and if so, on which side of t? What about if angle e is 40 degrees in measure?
5. How can you tell, from measuring angles b and e, whether the lines m_1 and m_2 intersect, and if so, in which half plane? Apply the principles of the previous section.
6. State a general principle whereby you can say that m_1 and m_2 are parallel when you know the measures of a pair of alternate interior angles formed by the two lines and a transversal t.

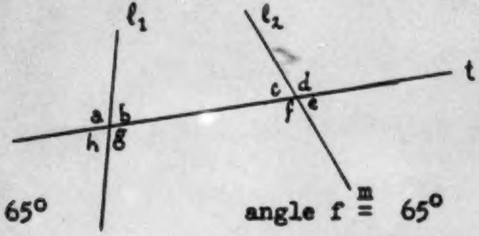
Principle 4. ____.

Use principle 2 as a model in trying to write your own statement of principle 4. After comparing your statement with those of your classmates, discuss the matters one should think about when one is trying to state a mathematical proposition.

7. Now try to state a principle by which you can tell on which side of the transversal the lines m_1 and m_2 will meet, from knowing the measures of a pair of alternate interior angles.

Principle 5. ______________________________.

8. You can sometimes discover where improvements are needed in a statement when you try to apply it to problems. Predict in each of the following cases whether the lines ℓ_1 and ℓ_2 will intersect and if so, on which side of t, and tell which principles you used.



a. angle b $\overset{m}{=}$ 65° angle f $\overset{m}{=}$ 65°

b. angle g $\overset{m}{=}$ 60° angle e $\overset{m}{=}$ 60°

c. angle a $\overset{m}{=}$ 50° angle c $\overset{m}{=}$ 40°.

d. angle h $\overset{m}{=}$ 90° angle c $\overset{m}{=}$ 100°

e. angle g $\overset{m}{=}$ 130° angle d $\overset{m}{=}$ 50^c

f. angle b $\overset{m}{=}$ 50° angle c $\overset{m}{=}$ 120°

g. angle a $\overset{m}{=}$ 130° angle d $\overset{m}{=}$ 70°

h. angle h $\overset{m}{=}$ 40° angle d $\overset{m}{=}$ 70°

i. angle g $\overset{m}{=}$ 60° angle c $\overset{m}{=}$ 50°

j. angle b $\overset{m}{=}$ 80° angle d $\overset{m}{=}$ 80°

9. Draw figures for each of these cases and check your predictions.

Exercises

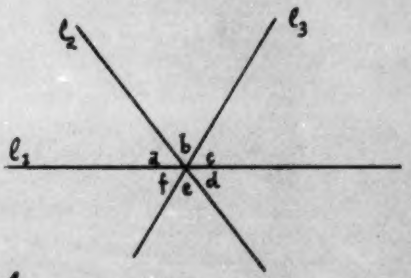
1. Make a list of the new kinds of angle pairs you have studied.

2. In the figure, name the following:

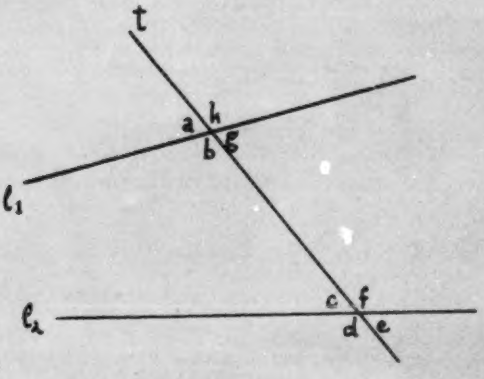
   a. Two pairs of vertical angles.

   b. Two pairs of adjacent angles.

   c. A set of three angles the sum of whose measures is 180 degrees.



3. In the figure, name:



   a. Two pairs of vertical angles.

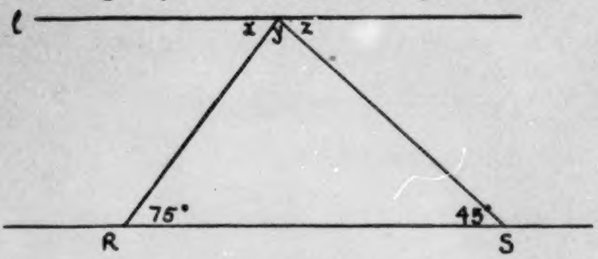
   b. Two pairs of adjacent angles.

   c. A pair of angles whose sum is 180 degrees.

   d. Two pairs of corresponding angles.

   e. Two pairs of alternate interior angles.

4. a. In this figure, draw the line ℓ parallel to R S.



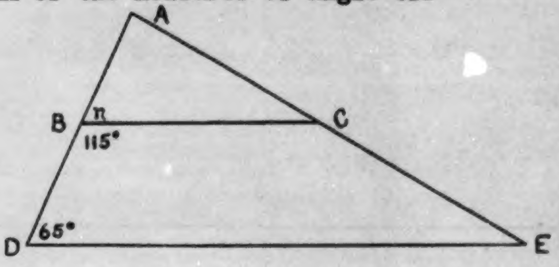
   What must the measure of angle x be in order that ℓ be parallel to R S? What principle did you use?

   b. What is the measure of angle z? Why?

   c. What is the sum of the measures of the angles x, y, and z?

   d. What is the measure of angle y?

5. a. In the figure, what is the sum of the measures of angle CBD and angle BDE?



   b. What is the measure of angle n?

   c. Is BC parallel to DE? Why?

6. Cut a triangle out of cardboard, making the sides of different lengths. Then draw parallel lines by (1) placing one side of the triangle along the edge of your paper; (2) drawing a line along

another side of the triangle; then (3) sliding the triangle along the edge of the paper; and (4) drawing a line along the same side of the triangle.

b. What principle is illustrated?

Part E. TURNING A STATEMENT AROUND (Converse)

We have seen that if certain things are true, _then_ certain other things are true. For example,

1. _If_ two angles are vertical angles, _then_ the angles have the same measure. Suppose we make a new statement, by interchanging the "if" part and the "then" part. This is the new statement:

   _If_ two angles have the same measure, _then_ the angles are vertical angles. The new statement, obtained by interchanging the "if" clause with the "then" clause, is called a _converse_ of the first statement.

   We have found that the first statement is true. Is the second statement always true? Must any two angles which have the same measure be vertical angles? If your answer is "No", then make a drawing of a case in which this statement is false.

Look at this statement.

2. _If_ Mary and Sue are sisters, _then_ Mary and Sue have the same parents.

   Is this statement true?

   Let us make the converse of this statement, by interchanging the "if" clause and the "then" clause.

   _If_ Mary and Sue have the same parents, _then_ Mary and Sue are sisters.

   Is this statement also true? Yes, it is true.

   We can see from these two illustrations that, if a statement is true, its converse, obtained by interchanging the "if" part and the "then" part, may be true or may be false.

3. Look at the statements below, and tell whether each one is true. Then write the converse, and tell whether the converse is necessarily true.

   a. If Rover is a dog, then Rover has four feet.
   b. If Blackie is a dog, then Blackie is a cocker spaniel.
   c. If a figure is a circle, then it is a closed curve.
   d. If the time is 10 A. M., Mary should be in school.
   e. If a figure is a closed curve composed of three line segments, then the figure is a triangle.

4. Make a table showing which of the above statements are true and which of the converse are true.

5. Make up five true statements, then state their converses. Make up some so that the new statement is true, and some for which it is false.

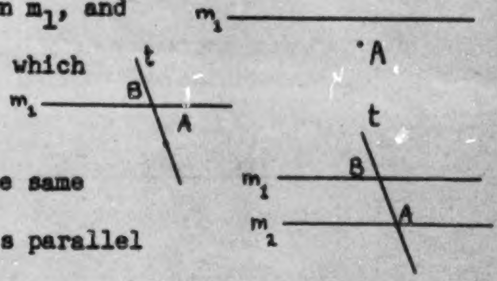
You can see from exercises 1-5 that when a true statement is turned around to form a converse, the converse is sometimes true and sometimes false.

Part F. CONVERSES OF PRINCIPLE 2 AND PRINCIPLE 4

Let us see whether the converse of Principle 2 seems to be true or false.

1. Draw a line m_1. Mark a point which is not on m_1, and call the point A. Through A, draw a line t, which intersects m_1 at point called B.

2. Using your protractor to make angles with the same measure, draw a line through point A which is parallel



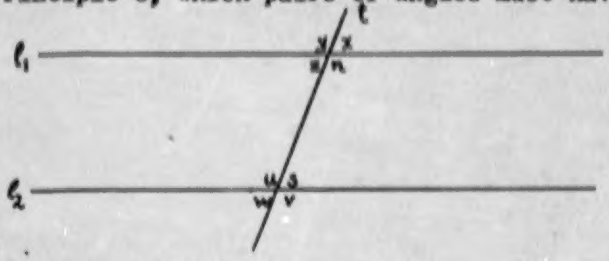
to line m_1. (What two kinds of angle pairs could you use?) Call this line m_2.

3. Now draw a second transversal intersecting m_1 and m_2. Call it t_1. Measure a pair of corresponding angles along transversal t_1. Are the measures of the corresponding angles the same?

4. Compare your results with those of your classmates. Do your angles have the same measures as theirs? What is true in each case? State your conclusion in the form of a general principle which tells the relation between corresponding angles formed when a transversal intersects two parallel lines.

Principle 6. ____.

5. In the figure below, suppose that line ℓ_1 is parallel to line ℓ_2. According to Principle 6, which pairs of angles must have the same measure?



Ex. 5-7

6. What other angle pairs will then have the same measure? Why?

7. If ℓ_1 is parallel to ℓ_2 then angle z $\overset{m}{=}$ angle w. Why?

   Angle w $\overset{m}{=}$ angle s. Why?

   Then angle z $\overset{m}{=}$ angle s. Why?

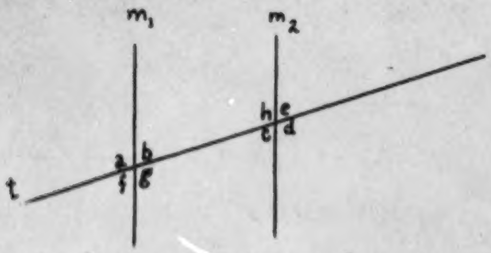
   What kind of angle pairs are angle z and angle s? Why?

8. Can you state a new principle from the reasoning you have done in Exercise 7?

Principle 7. ____.

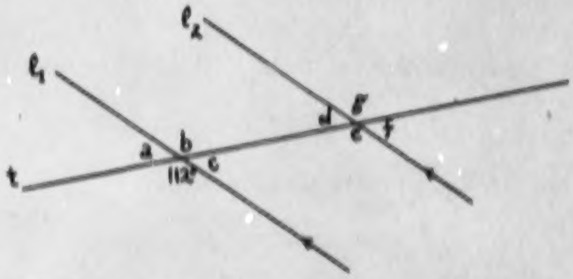
9. In the figure below, if line m_1 is parallel to line m_2 and a transversal t intersects them, which angle-pairs have the same measure because of Principle 7?

10. What other angle-pairs have the same measure? Why?

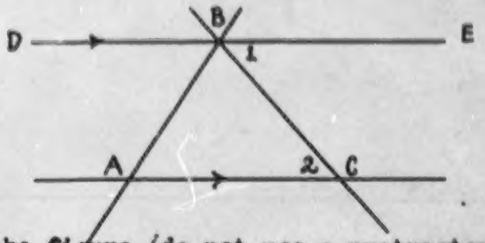


Ex 9 - 10

11. In the figure, if ℓ_1 is parallel to ℓ_2 and the measure of one angle is 112 degrees, as shown, what is the measure of each of the other angles? (The arrows on ℓ_1 and ℓ_2 remind us that those are parallel lines.)



12. In this figure, ABC is a triangle, and DE is a line on point B. Also, DE is parallel to AC. How many transversals intersect DE and AC?

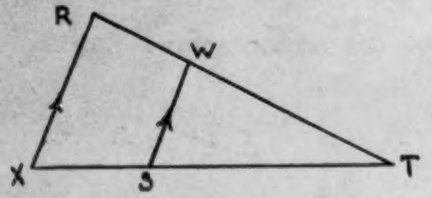


Ex. 12 - 14

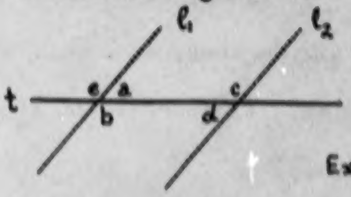
13. Make a sketch of the figure (do not use a protractor), but leave out line AB. What kind of angle pairs are angle 1 and angle 2?

14. Make another drawing of the figure, and this time leave out BC. What kinds of angle pairs do you see? Name the same angle pairs in the figure with all lines left in. (Use the three letter way of naming angles.)

15. In the figure below, suppose WS is parallel to RX. How many transversals intersect RX and WS? Find a pair of angles which have the same measure. Can you find a second pair?



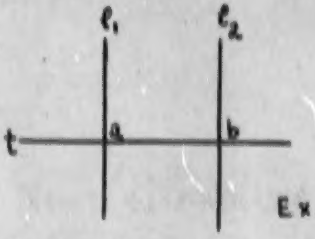
16 In this figure, ℓ_1 is parallel to ℓ_2 and t is a transversal. What is the sum of the measures of angles a and b? Why? What is true of angle b and angle c? Why? Then what is true of angle a and angle c? Would the same thing be true about angle b and angle d?



Ex. 16-17

17. Can you prove that angle a + angle c $\overset{m}{=}$ 180 degrees by using angle e instead of angle b? Explain your reasoning.

18. In this figure, if ℓ_1 is parallel to ℓ_2, and t is perpendicular to ℓ_1, what else must be true of ℓ_2 and t?



Ex. 18-19

19. Can you give reasons for these statements about the figure?

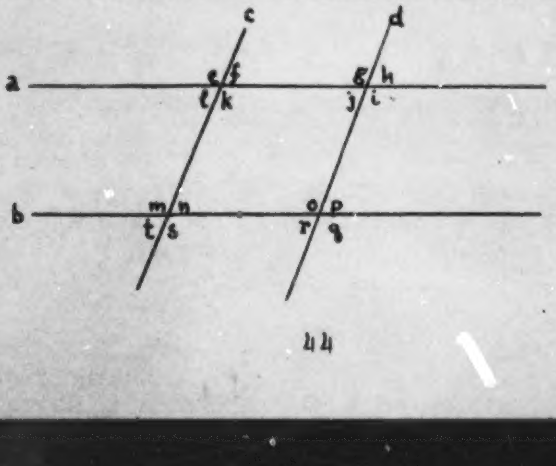
a. If t is perpendicular to ℓ_1, angle a $\overset{m}{=}$ 90 degrees. Why?

b. If ℓ_1 is parallel to ℓ_2, angle b $\overset{m}{=}$ angle a. Why?

c. Then angle b $\overset{m}{=}$ 90 degrees. Why?

d. Then t is perpendicular to ℓ_2. Why?

20. Here is a picture of two sets of railroad tracks which cross.



a. How many transversals do you see? Tell which lines each transversal intersects. Which lines are parallel?

b. Make a list of all pairs of corresponding angles.

c. List all pairs of alternate interior angles.

d. List all pairs of vertical angles.

e. List all pairs of non-adjacent angles whose measures have a sum of 180 degrees.

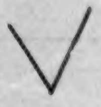
f. Suppose the measure of angle e is 152 degrees. Find the measures of as many other angles as you can.

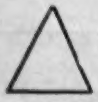
PART G. TRIANGLES

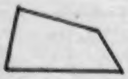
You have been discovering angle relations in a figure composed of three lines, two parallel lines and a transversal. Suppose the two lines are not parallel, so that each line is a transversal intersecting the other two lines. How does such a figure look?

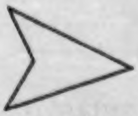
The three lines intersect in three points and, as you know, three points (not on the same line) and the segments joining them in pairs are a triangle.

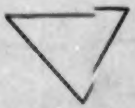
1. Which of the figures below are triangles? If a figure is not a triangle, tell which requirement of the definition it lacks.

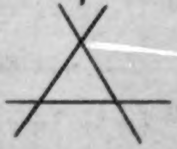
a


b


c.


d


e


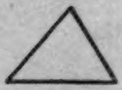
f


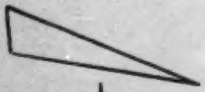
2. Draw a triangle which has two sides with the same measure. Such a triangle is called an isosceles triangle.

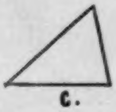
3. Draw a triangle which has three sides with the same measure. It is called an equilateral triangle.

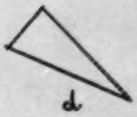
4. Draw a triangle with no two sides having the same measure. It is called a scalene triangle.

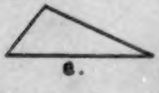
5. Which of the triangles below appear to be equilateral? Scalene? Isosceles?


a.


b.


c.


d.


e.

6. Draw two equilateral triangles, two isosceles triangles and two scalene triangles.

7. If a triangle is equilateral, is it also an isosceles triangle?

8. If possible, draw triangles whose sides have these measures: 3, 4, 5; 2, 7, 8; 1, 6, 7; 1, 6, 9. Which of these measures are not possible for the sides of a triangle? Why?

9. How can you tell whether a triangle can be formed if you know the measures of the three line segments?

If the distance from the point A to the point B is five inches, and the distance from B to C is two inches, what is the greatest possible measure for the distance from A to C? What is the least possible measure for this distance? Draw figures to illustrate both extreme cases and several intermediate cases.

If A, B, and C are the vertices of a triangle, then the angles ABC, BCA and CAB are called the angles of the triangle. The triangle is often called the triangle ABC.

10. Draw a large isosceles triangle ABC, making the sides AB and BC equal in length, but different from AC. Cut out the triangle carefully from paper, and then fold it so that AB lies exactly on CB. What do you notice about the angles CAB and ACB? Is one larger than the other or do they fit exactly, too? Try this experiment

with a number of isosceles triangles. Compare your results with those of your classmates.

State a general principle about the relation between the measures of the angles opposite two sides of a triangle which are equal in length.

Principle 8. ____.

Do at least five experiments like this, constructing triangles with two angles equal in measure. Compare your results with those of your classmates.

State a general principle about the relation between the lengths of the sides opposite two angles of a triangle which are equal in measure.

Principle 9. ____.

11. Draw a line segment about 4 inches long, and at each end measure an angle with the same measure. About 40 degrees is a good size. Extend the sides of the two angles until they intersect to form a triangle. Cut out the triangle and fold it so that the one angle fits exactly over the angle which has the same measure. Do the sides fit exactly too?

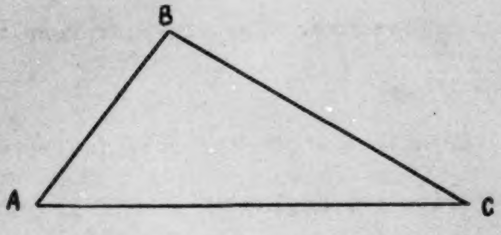
12. Use the relation you observed in exercise 11 to show that if a triangle has three sides equal in length, then all three angles are equal in measure also.

13. Use the result of exercise 12 to show that if a triangle has three angles equal in measure, then it has three sides equal in length also.

14. Any two equilateral triangles have the same shape, that is, one may be thought of as a picture of the other, but on a smaller scale.

Is the same thing true of all isosceles triangles? Draw several isosceles triangles to illustrate your answer.

15. In exercises 10-11 you investigated the relations when the measures of two sides or two angles of a triangle are equal. Now you may explore these relations in scalene triangles. Make at least ten triangles with unequal sides and measure their angles. Record your results in a table like this:



BC	∠ CAB	AB	∠ BCA	AC	∠ ABC

Give the measures of the sides in inches and of the angles in degrees.

A good scientist arranges his observations systematically. You have here 6 quantities which may have various values. In order to discover the relations between them, it is best to hold several of them fixed, vary one of the others, and measure the rest. You will probably find it convenient to choose values for the lengths of BC and AB and keep these the same throughout your experiment. Then choose different values for the length of AC. Do the lengths of the sides determine the size and shape of a triangle completely? Then measure the angles, and record your observations as above. Your results in exercises 8-9 should give you information about the possible choices for the length of AC.

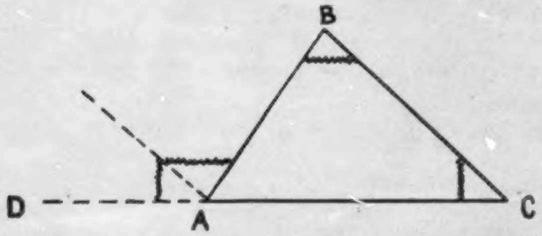
Notice that the column for each angle is next to that for the opposite side.

You and your classmates should each choose different lengths for BC and AB. In this way you will have plenty of experimental data for comparison.

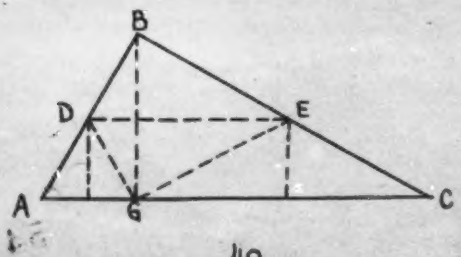
On the basis of your observations you should arrive at certain conclusions. When one side is longer than another, how do the measures of the opposite angles compare? Is there a simple relation between the lengths of the sides and the measures of the angles? If you double the length of AC, does the measure of ∠ ABC change in a simple way? What happens if you double the lengths of all three sides of a triangle?

Part H. ANGLES OF A TRIANGLE

1. Draw a triangle ABC and cut it out of paper. Tear off the corners at B and C and mount the whole figure on cardboard as shown, with the cut out corners pasted in around the vertex A. Measure the angle DAC. Compare the result of your experiment with those of your classmates. Is there a general law which fits your observations?



2. Cut a triangle ABC out of paper and mark off the midpoints D and E of the sides AB and BC respectively. (The midpoint is halfway

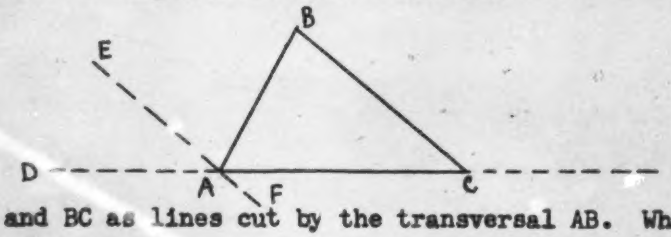


from one endpoint to the other, so that AD $\overset{m}{=}$ DB, BE $\overset{m}{=}$ EC.) Fold over the segment AD so that A lies over some point G of the line AC. Similarly, fold over EC so that C lies over G. Finally, fold along the segment DE. Where does the point B fall? Where do the segments DB and EB lie? What is the sum of the measures of the angles of the triangle? Do this experiment with other paper triangles. Does the law you found in exercise 1 fit these observations? State your law in the form of a principle concerning the sum of the measures of the angles of a triangle.

Principle 10. ____.

You have now discovered a certain law by experiment. Were these experiments necessary? Is there any connection between this principle and the ones you found before? We shall now try to help you discover a proof of principle 10 from the previous ones. In other words, if you know the other principles, then you can arrive at this one by logical reasoning without a single additional experiment.

3. The experiment of exercise 1 suggests one line of reasoning. Instead of tearing off corners, simply draw a line AE so that Angle EAB has the same measure as angle ABC. Extend the line AC.



Think of AE and BC as lines cut by the transversal AB. What kind of angle pair are angles EAB and ABC? What conclusion can you draw from principle 4?

Now think of AE and BC as lines cut by the transversal AC. What kind of angle pairs are angles DAE and ACB? Does either principle 6 or 7 apply? What is the sum of the measures of the angles DAE, EAB, and BAC? What do you conclude about the sum of the measures of the angles of triangle ABC?

4. Instead of working with the angle DAE, we could have compared angle ACB with angle FAC. Which principle applies in this case? Write out the complete proof from beginning to end, making any necessary changes, and giving the reason for each step of your proof.

   Compare your proof with those of your classmates. Did you omit any necessary steps? Did you have a convincing reason for every step? Was each statement in your proof clear and precise? Was your conclusion stated in such a way that everyone else understood exactly what you proved?

   Discuss with your classmates the things which make a proof good. Are there any common faults or mistakes which you must watch out for? After the class discussion rewrite your proof in accordance with the points brought out by you and your classmates.

5. In each of the following cases the measures of certain parts of the triangle ABC are given, those of the sides in inches and those of the angles in degrees, and you are asked to predict the measure of some other part. In each case, give your reason.

	Given	To Find	Principle
a.	$\angle ABC \overset{m}{=} 60^\circ$, $\angle BCA \overset{m}{=} 40^\circ$	$\angle$ CAB	
b.	$\angle ABC \overset{m}{=} 90^\circ$, $\angle CAB \overset{m}{=} 20^\circ$	$\angle$ BCA	
c.	$\angle CAB \overset{m}{=} 20^\circ$, $\angle BCA \overset{m}{=} 30^\circ$	$\angle$ ABC	
d.	$\angle BCA \overset{m}{=} 110^\circ$, $\angle ABC \overset{m}{=} 40^\circ$	$\angle$ CAB	
e.	$\angle ABC \overset{m}{=} 52^\circ$, $\angle BCA \overset{m}{=} 37^\circ$	$\angle$ CAB	
f.	$\angle ABC \overset{m}{=} 40^\circ$, AB $\overset{m}{=}$ 2 in., AC $\overset{m}{=}$ 2 in.	$\angle$ ACB	
g.	AB $\overset{m}{=}$ 3 in., BC $\overset{m}{=}$ 3 in., CA $\overset{m}{=}$ 3 in.	$\angle$ BCA	
h.	$\angle BAC \overset{m}{=} 70^\circ$, $\angle BCA \overset{m}{=} 70^\circ$, AB $\overset{m}{=}$ 4 in.	BC	
i.	$\angle BAC \overset{m}{=} 100^\circ$, $\angle BCA \overset{m}{=} 40^\circ$, AB $\overset{m}{=}$ 4 in.	AC	
j.	$\angle ABC \overset{m}{=} 60^\circ$, $\angle CAB \overset{m}{=} 60^\circ$, AC $\overset{m}{=}$ 3 in.	AB	

Check your answers by drawing figures and measuring.

6. Suppose one angle of an isosceles triangle has a measure of 50°. Find the measures of the other two angles of the triangle. Are two different answers possible?

7. In triangles ABC and DEF, suppose that $\angle BAC \overset{m}{=} \angle EDF$ and $\angle BCA \overset{m}{=} \angle EFD$. What will be true about angles ABC and DEF? Why?

Informal Geometry I

Statement of Principles

Principle 1. When two lines intersect, the angles in each pair of vertical angles formed have equal measures.

Principle 2. When a transversal intersects two lines in the same plane, and a pair of corresponding angles have the same measure, then the lines are parallel.

Principle 3. When a transversal intersects two lines in the same plane and, of a pair of corresponding angles, the exterior angle has the larger measure, then the lines intersect on the same side of the transversal as these two angles. If, however, the interior angle of the pair has the larger measure, then the lines meet on the opposite side of the transversal from this pair of angles.

Principle 4. When a transversal intersects two lines in the same plane, and a pair of alternate interior angles have the same measure, then the lines are parallel.

Principle 5. When a transversal intersects two lines in the same plane, and one of a pair of alternate interior angles is smaller than the other, then the lines intersect on the same side of the transversal as the smaller angle.

Principle 6. When a transversal intersects two parallel lines, then each pair of corresponding angles have the same measure.

Principle 7. When a transversal intersects a pair of parallel lines, then each pair of alternate interior angles have the same measure.

Principle 8. If two sides of a triangle are equal in length, then the opposite angles are equal in measure.

__Principle 9__. If two angles of a triangle are equal in measure, then the opposite sides are equal in length.

__Principle 10__. In any triangle the sum of the measures of the angles is 180 degrees.

UNIT IX

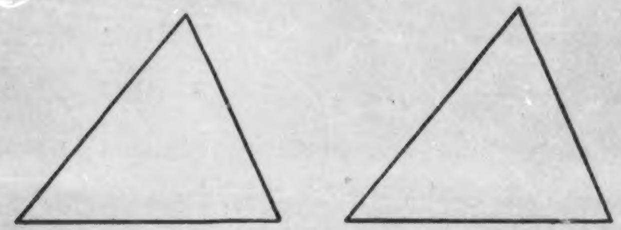
INFORMAL GEOMETRY II

Part A. CONGRUENT TRIANGLES

When leaves are drifting to the ground in the fall, have you ever tried to find two leaves from the same tree which have exactly the same size and shape? While we can tell whether a leaf is from an oak tree or a maple tree or some other kind of a tree, it is unusual to find two oak leaves which seem to be exactly the same size and shape if we place one on top of the other to compare them. You may also have wondered why, among the faces of all the people whom you know, or even among all the people in a great city, you so seldom find two which are very much alike.

The wheat fields of two farmers may be "exactly" the same shape and size, although they usually are not. A manufacturer of large bolts of a certain kind employs methods of "quality control" to try to make sure that the bolts he makes will be as nearly alike as possible. Mathematics is used to determine whether objects that appear to be alike actually are, and to determine the amounts and importance of difference in the objects.

In order to study the characteristics of objects that are exactly alike, we begin, as scientists usually do, with a very simple situation. It turns out that the case of two triangles is the key to the general problem of determining when two objects have exactly the same size and shape.



These triangles appear to be the same size and shape; if we cut one of the triangles out of the paper, and find that we can place it on the other so that the three vertices and sides of one triangle lie exactly on the three vertices and sides of the other triangle, we say that the two triangles are congruent. Or, if we could move one of the triangles in its plane until it fits exactly on the other, we would say that the triangles are congruent.

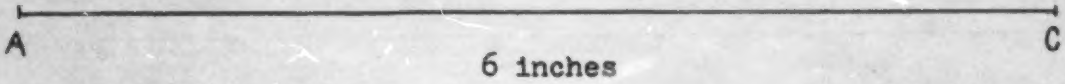
Principle 1: Corresponding segments and angles of congruent figures have equal measures.

If one triangle were cut out of a rubber sheet, then we could stretch or shrink parts of it while we move it, and thus make it fit on the other. In such a case we would certainly not call the triangles congruent. It is essential that the motion of the triangle be rigid, that no distances between points of the figure should change during the motion.

We could compare solid figures in a similar way by making a mold to fit one solid, and then testing the other solid by trying the mold on it. Again, in this process we must be sure that, when we move things around, the motions are rigid.

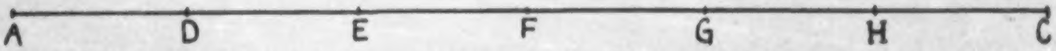
Now let us look more carefully at this method of comparing figures, and see what it really means. If we understand how it works, we may be able to discover a method of comparison which does not require a physical motion. When we fit one figure onto another, we make each point of the first figure correspond to a certain point of the other, point for point. The distance between any two points of the first figure is the same as the distance between corresponding points of the other. Since we bring the first figure to fit on the other by a rigid motion, during which no distance is changed, then the points of the original figures can be matched with each other so that the distance between any pair of points in the first figure is the same as the distance between the corresponding pair of points in the other.

For example, draw triangles A B C and A' B' C' so that the lengths of A B and A' B' are 4 inches, those of B C and B' C' are 5 inches, and those of C A and C' A' are 6 inches. (How can you draw a triangle with sides of given length? Suppose you locate A and C 6 inches apart. Where should B be?



Forget about the length of B C for a moment and concentrate on the requirement that A B should be 4 inches long. Where do the points lie whose distance from A is 4 inches? Do you have any instrument for drawing the figure formed by these points? Next draw the curve formed by all points whose distance from C is 5 inches. Where are the possible positions for B? How many are there?) Cut out triangle A B C and see if this triangle is congruent to A' B' C' by fitting it onto the latter.

Now separate the triangles again. Using your ruler, mark off on the side AC points D, E, F, G, and H, so that



the distances from A are 1, 2, 3, 4, and 5 inches respectively, and mark off on the side A B points J, K, and L whose distances from A are 1, 2, and 3 inches respectively. Can you predict the positions of these points when you place the triangle A B C on A' B' C' so that A lies on A', B on B' and C on C'? Mark off your predictions on the triangle A' B' C' and label them D' E' F', . . ., L'. Now check your predictions.

Measure the distance between the various pairs in the first figure and record them in a table:

	A	B	C	D	L
A	0	4	6		
B		0	5		
C			0		
D					
L					

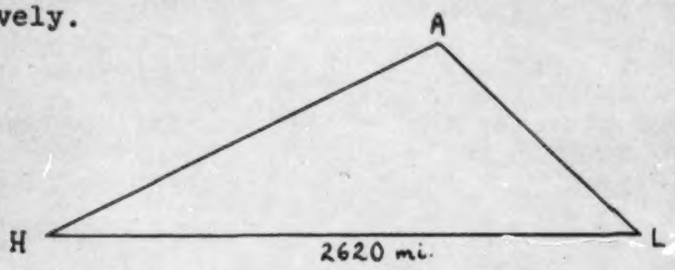
Make a corresponding table for the points A', B', . . ., L'. Compare distances between corresponding points.

Pick a point M in the interior of the triangle A B C. How can you predict the position of the corresponding point M' in the other triangle? Are there any lines which you can draw and distances which you can measure so that you could locate M' exactly? Make your prediction and check it by placing the triangle A B C upon the triangle A' B' C'. Does M lie above your prediction?

If we know the distance between every pair of points in a figure, this gives us complete information about its size and shape. In the above example of the triangle A B O, you can find as many points as you please on the three sides. You could measure all distances and give a complete description of the triangle. Is all this information necessary? If you know the lengths of the sides, and you locate the point D on A C by marking off the length of A D as 1 inch, does this information determine the distance from B to D?

Is more than one value for this distance possible? Incidentally, if D is on the side AC, and AD = 1, AC = 6, then what is the length of DC? How can you tell from the lengths of AD, DC, and AC whether D is on the segment AC? Then is there more than one possibility for the length of BD?

Sometimes you may know the sizes of certain angles and certain segments. Imagine that you are at point A in the figure below, and you are the navigator of an airplane. You receive signals from the radar operators at the airports in Los Angeles and Honolulu which tell you that the measures of the angles AHL and ALH are 20° and 30° respectively.



Your chart shows that the distance from Honolulu to Los Angeles is 2620 miles. Is this enough information for you to be able to determine your distance from Honolulu? Make a drawing to scale using 1/8 of an inch to represent 100 miles. How long should HL be? Is there more than one possible length for AH? Measure AH, and calculate the distance from your airplane to Honolulu.

The main problem of this unit is to find what parts of a figure must be known in order to determine all the other parts. We could also think of the problem as that of investigating what parts of two figures should be measured in order to know whether they are congruent. This would enable us to find whether figures are congruent without actually moving them. We have already obtained some evidence that it is not necessary to measure all parts of a triangle in order to determine the complete size and shape.

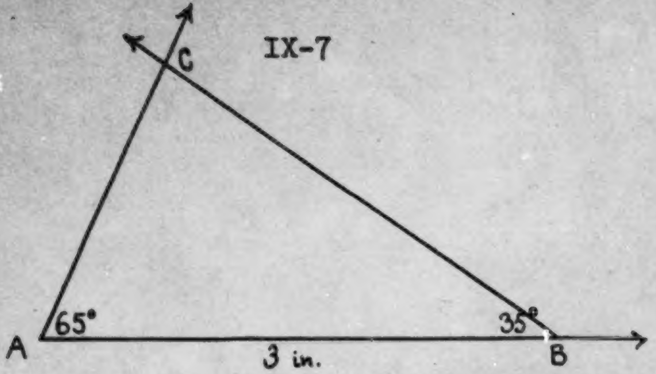
Before we tackle the general problem of when two figures are congruent, we start, like good mathematicians, with the simplest case, namely that of two triangles. After we thoroughly understand this case, we will be prepared to study more complicated figures.

A triangle has six parts -- three sides and three angles. We shall investigate which combinations of these parts are enough to determine the rest, and so to determine the size and shape of the the triangle.

Choose a point A and a ray with A as end point. Find a point B on the ray whose distance from A is 3 inches. How many possible positions are there for the point B? Choose one of the sides of the line AB and, using a protractor, draw a ray on this side with A as endpoint which makes with AB an angle whose measure is 65°. On this same side of AB draw another ray with B as endpoint which makes with AB an angle whose measure is 35°. Do these rays intersect? If so, label the intersection C. Once you have chosen the point A and drawn the ray on which B should lie, is there more than one possible way of drawing the triangle ABC according to the above directions?

Try this experiment again on another sheet of paper. Is the second triangle congruent to the first? Try this for three separate triangles, each time making the line segment equal in length to AB and drawing the angles with the measures and positions shown in the figure. Cut out two of the triangles and place them on the third one. What do you conclude?

In the figure, AB is the side included between the angle whose measures are 35 degrees and 65 degrees. To be more brief, we say, "two angles whose measures are 35 degrees and 65 degrees and the included side."



EXERCISES

1. Draw two triangles making one side of each triangle 3 inches long, and the angles on the sides with measures 40 degrees and 80 degrees. Measure the other two sides of each triangle and compare the lengths. What do you conclude?

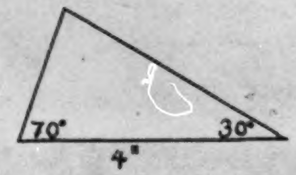
2. If two angles of a triangle measure 40 degrees and 80 degrees, what is the measure of the third angle?

3. Draw two triangles each of which has a side 3 inches long, and angles which measure 40 degrees, 80 degrees, and 60 degrees. Draw the 3 inch side between the vertices of the 40 degree and 60 degree angles. Cut out these triangles and compare them with each other, and with the triangles you drew in exercise 1. What do you find?

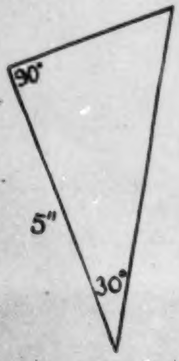
4. Draw two triangles, each of which has angles which measure 40 degrees, 80 degrees, and 60 degrees. In your drawing make the side between the vertices of the 60 degree and 80 degree angles 3 inches long. Cut out the triangles and compare them with each other, and with the triangles you drew for exercises 1 and 3. What do you find?

5. Exercises 3 and 4 should convince us that the following statement is not necessarily true: If two angles and one side of a triangle are equal in measure to two angles and one side of another triangle, the triangles are congruent. Correct the statement so it will be true.

6. Each of the triangles below is congruent to one of the others. Name the pairs of congruent triangles.



(a)



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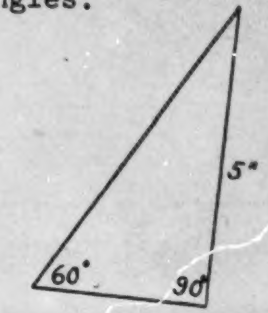

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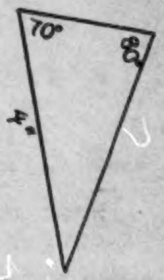
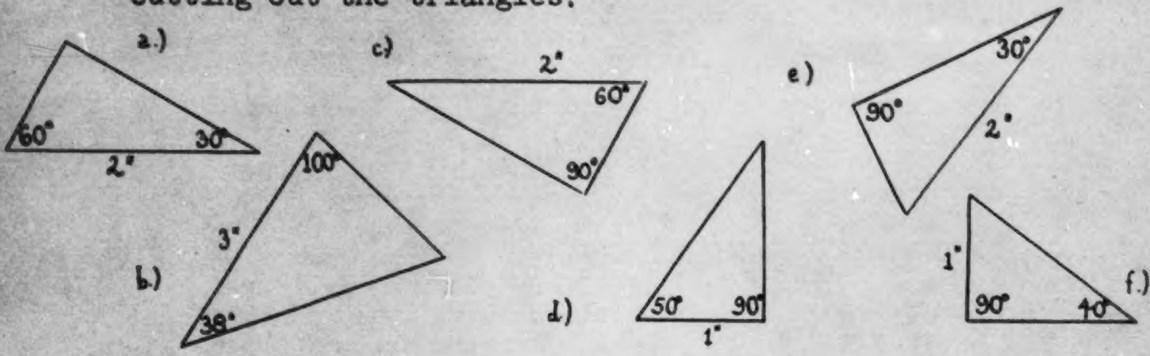
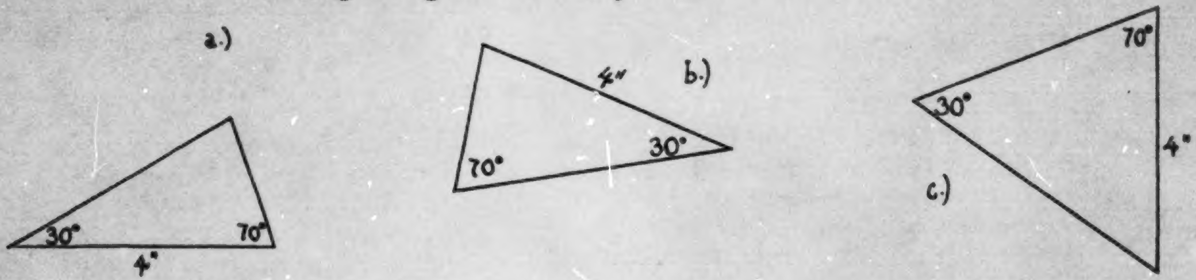


(d)

7. Group the following triangles into sets of congruent triangles. Can you do this without measuring any of the sides or angles, or cutting out the triangles?



8. Are the following congruent? Why?



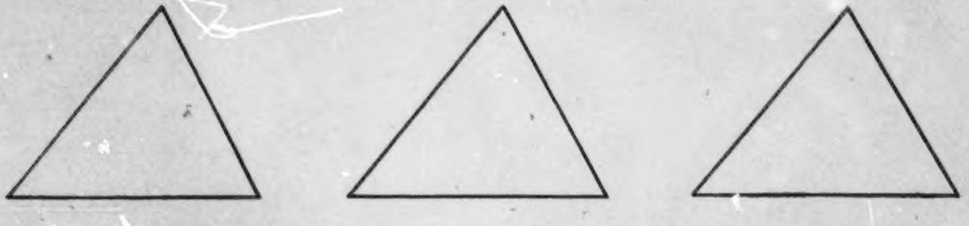
9. Draw a triangle with angles whose measures are 50°, 60°, and 70°. Can you draw a larger triangle whose angles are equal in measure to those of this triangle? Are the triangles necessarily congruent?

10. State the principle which you have found from the above exercises, whereby you can tell whether two triangles are congruent by measuring in each triangle two angles and the included side.

Principle 2:

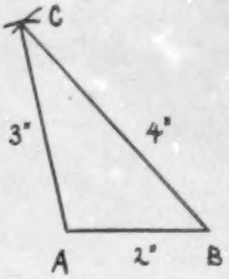
PART B. EXERCISES IN EXPLORING TRIANGLES

1. Compare with a compass the sides of the three triangles below:



What do you conclude? Do the three triangles appear to be congruent?

2. Draw a triangle with sides whose lengths are 2 inches, 3 inches, and 4 inches. This is rather hard to do with a ruler, but easy if you use a compass. Follow these directions: Draw a line segment 2 inches long. Call it AB. Spread your compass so the points are three inches apart. With the point of the compass on point A, draw part of a circle above AB. Spread the compass so the points are 4 inches apart. Place the point of the compass on point B, and draw part of a circle so it intersects the part of a circle you drew before. Call the point of intersection point C. Draw line segments AC and BC. Are the sides of the triangle ABC 2 inches, 3 inches, and 4 inches long?



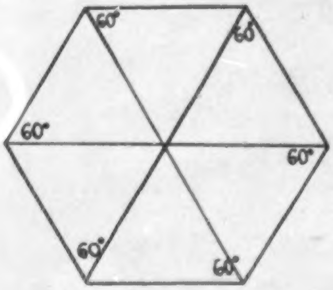
3. Draw a second triangle with sides 2 inches, 3 inches, and 4 inches long. Cut out the second triangle and test it to see whether it is congruent to the triangle you drew for exercise 2. What is your conclusion?

4. Draw three triangles with sides 2 inches, 2.5 inches, and 4 inches long. Test them to see whether they are congruent as you did before. What is your conclusion?

5. State a general principle whereby you can tell whether two triangles are congruent from measurements of their sides.

Principle 3:

6. Do you find any congruent triangles in this figure? On what do you base your answer?



7. Devise a method for constructing a triangle if you are told the measure of its three sides, using the following tools: pencil, paper, a ruler, 3 thumb tacks, and a piece of string.

PART C. CONDITIONS FOR CONGRUENCE OF TRIANGLES

Principle 2 and principle 3 state two cases in which two triangles are congruent if certain three parts of one triangle are equal in measure to similarly placed parts in another triangle. A complete listing of sets of three parts of a triangle is as follows:

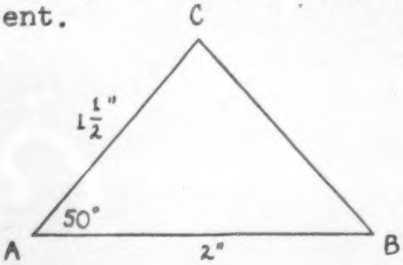
I. 3 angles
II. 3 sides
III. 2 angles and the included side
IV. 2 sides and the included angle
V. 2 angles and a side *not* included between the vertices of the angles
VI. 2 sides and an angle *not* included between the sides.

In principles 2 and 3 you have stated methods for deciding whether two triangles are congruent on the basis of measurements of the parts listed in III and II above. In exercise 9 on page 8 you investigated whether measurements of the parts mentioned in I are enough to enable you to tell whether two triangles are congruent. What was your conclusion?

Recall principle 10 of the previous chapter. Is it necessary to measure all the three angles of a triangle in order to know their sizes? Can you apply this idea to case V above? Try to use the mathematician's favorite trick of *reducing a problem to the previous case*.

EXERCISES.

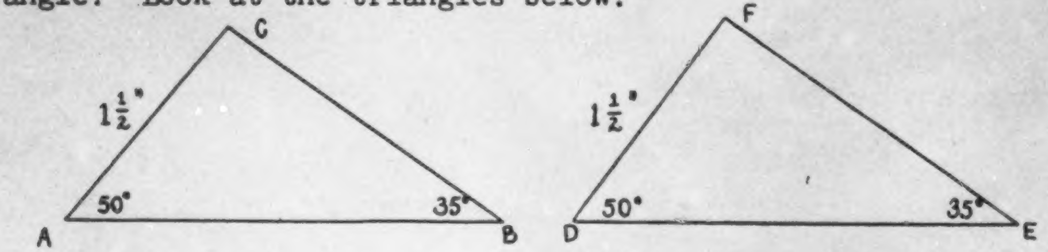
1. Consider the set of parts listed as IV, two sides and the included angle. The positions of the parts in this set are illustrated in the drawing below. Use the measures given and draw three triangles. Then test the triangles to see whether they are congruent.



If the point A, the ray AB, and the side of AB on which C lies are given, is it possible to draw more than one triangle with sides and angles having the above measures?

2. Consider the set of parts listed as V, two angles and a side which is not included between the vertices of the angles.

   If two angles of one triangle have the same measures as two angles of another triangle, what will be true about the third angle? Look at the triangles below.



3. Now consider Case VI. Draw a triangle ABC so that AB = 3 inches, angle A = 30°, and BC = 2 inches Let us see,how many possible shapes can such a triangle have? After you locate the points A and B 3 inches apart, then you still have to find the point C. Choose the side of the line AB where you want the triangle to lie. Where do all the points C lie whose distance is 2 inches from B? Draw the figure formed by this set of points. Now draw angle A making it 30°. How many positions are possible for C in this halfplane? What are the possible lengths for the side AC?

   Can you decide whether two triangles are congruent by measuring the parts in case VI?

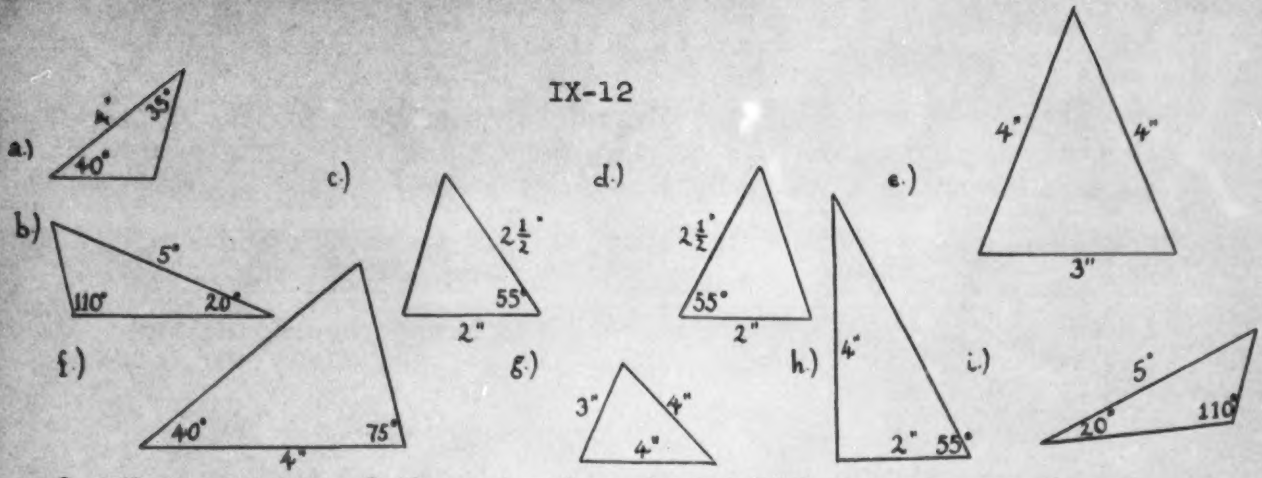
4. On the basis of your experiments in exercise 1, state a principle whereby you can decide whether two triangles are congruent by measuring the parts given in case IV.

Principle 4:

5. State your conclusions about the set of parts listed in V and the set of parts listed in VI.

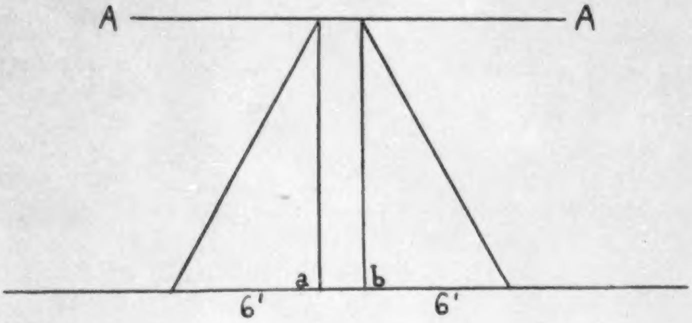
PART D. EXERCISES ON CONGRUENCES OF TRIANGLES.

1. For each of the triangles below, describe the set of parts whose measures are marked. You may use the descriptions listed on page 10.



2. Name any two of the triangles above which are congruent, and tell which principle applies.

PART E. PERPENDICULAR BISECTORS



1. The telephone pole in the figure is supported by two wires, each of which is fastened to the ground at a point 6 feet from the base of the pole. The Telephone Company knows that these guy wires will be about equal in length. How do they know this?

   A. The company has set up the pole so it is perpendicular to the ground. How large are angles a and b?

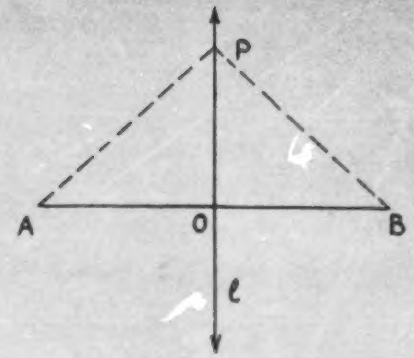
   B. In the two triangles the lines represented by the sides of the pole are equal in length.

   C. The sides of the triangles along the line representing the ground are each 6 feet long.

   What can you conclude from statements A, B, and C? Which principle applies?

2. In exercise 1, suppose the company attaches two more guy wires to the pole in the same way as in exercise 1. What will be true about the lengths of these guy wires? Why?

In the figure the line ℓ is perpendicular to segment AB. The intersection of the sets of points, AB and ℓ, is the point O, and the measures of AO and OB are the same. So line ℓ is called the perpendicular bisector of AB. Where does the word bisector come from?



We can show that if P is any element of the set ℓ then the length of PA is equal to the length of PB. In order to show this consider the triangle POA and triangle POB.

$$AO \overset{m}{=} OB$$

$$\text{Angle } AOP \overset{m}{=} 90 \text{ degrees}$$

$$\text{Angle } BOP \overset{m}{=} 90 \text{ degrees}$$

$$OP = OP$$

Therefore triangle AOP is congruent to triangle BOP, and PA $\overset{m}{=}$ PB.

EXERCISES

1. a. In the argument above, why does AO $\overset{m}{=}$ OB?
   b. Why does angle AOP $\overset{m}{=}$ angle BOP?
   c. Why is triangle AOP congruent to triangle BOP?
   d. Why, then, is PA $\overset{m}{=}$ PB?

2. Select a different element of the set ℓ and call it P. Would the argument above be changed in any way?

3. Consider the half-planes determined by the line on segment AB. If P is located on ℓ in the other half-plane than the one shown in the figure will you need to change the argument in any way? Make a drawing to illustrate your answers.

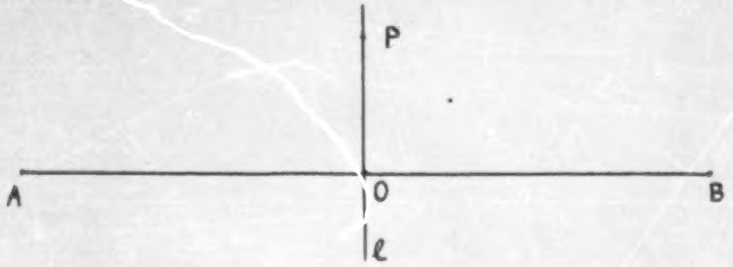
4. Draw a line segment. With ruler and protractor draw the perpendicular bisector of the segment.

5. a. Draw a line segment, and call it PR.
   b. Use ruler and protractor to draw the perpendicular bisector of PR. Call the perpendicular bisector 1 and its intersection with PR point A.

   c. Select an element of ℓ, and call it B.

5. d. Measure the distance from B to P and from B to R.
   e. What do you find is true? Is your conclusion the same as before?

6. Mark two points A and B on a sheet of tracing paper and draw the segment AB with a ruler. Fold over the paper so that B falls on A and the segment AB is folded over itself, and flatten the paper so that a crease is formed. Open it up and draw a line ℓ along the crease. Call the intersection of ℓ with AB the point O, and choose any other point P on ℓ.

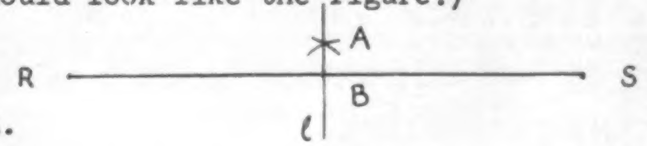


Since, when the paper is folded, the segment BO lies over the segment AO, what is true about the measures of these segments? Since the angle BOP lies over the angle AOP, what do you know about the measures of these angles? What is the sum of their measures? Can you tell what the measures of these angles must be without measuring them? Draw the segments PA and PB. If you fold the paper along ℓ, where will PB lie? How are the lengths of PA and PB related?

7. State a principle about the distances from any point on the perpendicular bisector of a line segment to the endpoints of the segment.

Principle 5:

8. Draw a line segment, and call it RS. Then use a compass to locate a point, not on RS, which is the same distance from points R and S.
   (Your drawing should look like the figure.)



Call the point A.

   b. Use your ruler to find the midpoint of RS. Call the midpoint B. Then draw AB.
   c. Does segment AB seem to be perpendicular to segment RS?
   d. Draw the line on AB and call it ℓ. Draw segments AR and AS.

9. In your figure for exercise 8,

$AR \overset{m}{=} AS$. Why?

$BR \overset{m}{=} BS$. Why?

$AB \overset{m}{=} AB$. Why?

So triangle ABR is congruent to triangle ABS. Why? Then how large are angle ABR and angle ABS? What is the sum of these angles? Does this show that ℓ is the perpendicular bisector of RS?

10. Repeat exercise 8 locating a point A in the other half-plane determined by the line of RS. Does this change the argument in any way?

11. State a principle whereby you can tell whether a point lies on the perpendicular bisector of a line segment by measuring its distances from the endpoints.

Principle 6:

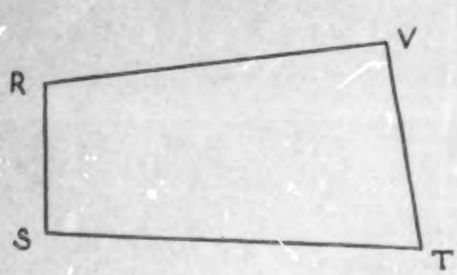
12. Compare the statements in exercises 7 and 11 above. What is the difference between them? Is one a converse of the other?

We can use principle 6 to guide us in construction of a perpendicular bisector with ruler and compass.

13. a. Draw a line segment, and call it DC.
    b. Use compasses to locate a point E the same distance from C and D.
    c. Use compasses to locate a second point F the same distance from C and D, in the other half-plane determined by the line on CD.
    d. Draw the line on E and F. Call it ℓ. Must E lie on the perpendicular bisector of CD? Must F lie on the perpendicular bisector of CD? Why? Is there any line on E and F, except ℓ? Then is ℓ the perpendicular bisector of CD?

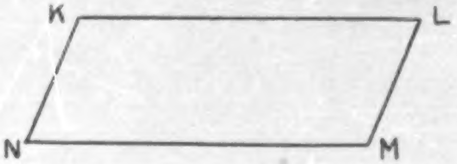
PART F. PARALLELOGRAMS

We call a figure with 3 points joined by 3 line segments a *triangle*. (Do you know what the prefix "tri" means? Think of the words like "triple", "trio", "tricycle", etc. Does this suggest the meaning of the word triangle?) We call a figure with 4 sides a *quadrilateral*. If R,S,T, and V are distinct points, then the figure formed by the segments RS,ST,TV, and VR is a quadrilateral, and is often called the quadrilateral RSTV. These 4 segments are its *sides*, and R, S, T, and V are called its *vertices*. The angles VRS, RST, STV, and TVR (also called "angles R, S, T, and V") are the angles of the quadrilateral.

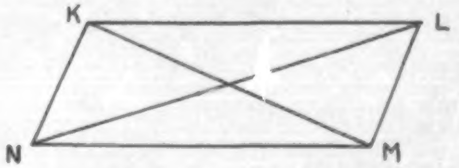


Since we have just made a rather thorough study of triangles, we are ready to attack problems about more complicated figures, and the next step is, of course, the study of quadrilaterals. Certain special kinds of quadrilaterals are important enough to be worth more detailed study.

A quadrilateral in which the pairs of opposite sides are parallel is called a parallelogram. This figure KLMN is a parallelogram.



The line segments, KM and LM, are called the diagonals of the parallelogram. The diagonals shown in the next figure.



By using the congruent triangles we can rather easily show that

a. If a quadrilateral is a parallelogram, then the opposite sides have equal measures.

b. If a quadrilateral is a parallelogram, the diagonals bisect each other.

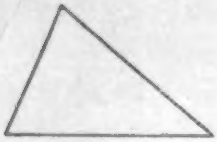
You could draw parallelograms and measure their sides and diagonals and hence conclude by experiment that these statements are probably true.

In the experimental method, however, we shall always have some uncertainty because the measurements are never exact, only approximate, and because we can at most test these statements in a large number of cases, so that we could not be sure that they are always true. Let us try to apply the deductive method instead.

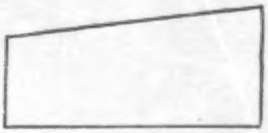
EXERCISES

1. Which of these figures are quadrilaterals?

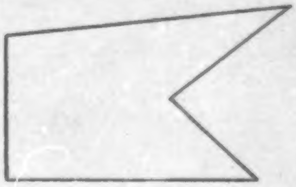
a.)



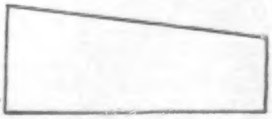
b.)



c.)



d)

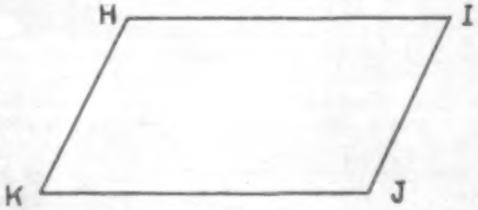


2. Which of the figures in exercise 1 appear to be parallelograms?

3. Draw two parallelograms of different sizes and by measurement, see if the statement (a) page 16 appears to be true.

4. Construct a parallelogram ABCD and cut it out. Cut the parallelogram along the diagonal AC. Are the triangles you obtained congruent? Check by fitting one onto the other. What sides and angles of the parallelogram have equal measures?

5. Let us try to help you discover for yourself why statement (a) is true. If you answer the following questions, referring to the parallelogram below, you may



be able to deduce the statement from the principles you already know.

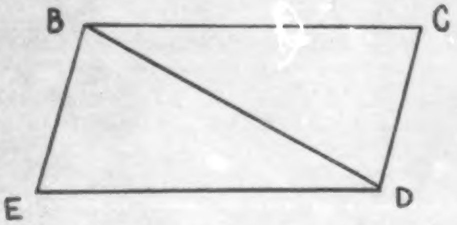
a. Which principles state that under certain circumstances two segments are equal in length?

b. Do any of them apply immediately to the above figure? If not, can you draw a segment in such a way that one of these principles might apply? Think of the preceding exercise.

c. What kind of lines are HI and JK? HK and IJ? How is the line you have just drawn related to these lines? Are there any angles which must have equal measures because of this?

d. Can you apply any of the principles of this chapter in order to show that certain triangles are congruent?

6. The figure BCDE below is a parallelogram.

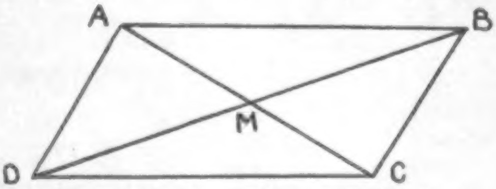


In the following reasoning, tell why the statement in each step is true.

a. BC is parallel to DE
b. $\angle$ CBD $\overset{m}{=}$ $\angle$ BDE
c. EB is parallel to CD
d. $\angle$ CDB $\overset{m}{=}$ $\angle$ EBD
e. BD $\overset{m}{=}$ BD.
f. The triangle BCD is congruent to the triangle DEB.
g. BC $\overset{m}{=}$ DE.

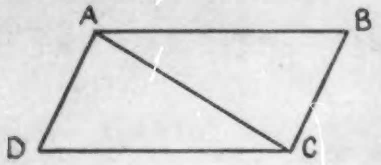
7. Draw two parallelograms of different sizes and, by making measurements, see if the statement b on page 16 appears to be true.

8. Show why the statement b. on page 16 is true. Give your reasons carefully. In the figure you might show that triangles AMB and CMD are congruent. If you think of alternate interior angles you can find angles with equal measures. Also statement a on page 16 may assist you in supporting this statement.



9. Draw a quadrilateral ABCD in which AB = DC and BC = AD. Measure the angles and apply one of the principles of the previous chapter. How are the opposite sides related? What special kind of quadrilateral is ABCD? Repeat this experiment several times. State a general principle which fits the results of your experiment.

10. In the figure below



$AB \overset{m}{=} DC$ and $BC \overset{m}{=} AD$. Prove that BC is parallel to AD by :
- (a) identifying congruent triangles;
- (b) telling why they are congruent;
- (c) showing what angles have equal measures as a result of the congruence, and
- (d) indicating why this enables us to conclude that the opposite sides of the quadrilateral are parallel.

Does your result agree with the principle which you discovered experimentally in exercise 9?

11. Make a statement by interchanging the if-then parts of statement (b) page 16.

12. Check whether or not the statement you have made in exercise 11 is probably true by drawing several quadrilaterals in which the if-part of that statement is true and measuring other parts.

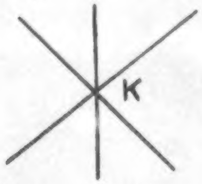
13. A square is a special kind of parallelogram. The measures of all sides of a square are equal and all angles have a measure of 90 degrees. Show whether or not the statement is true: The diagonals of a square are perpendicular to each other. Can you use Principle 6?

14. A rectangle is a special kind of parallelogram in which the angles are right angles. Show whether or not the following statement is true in a rectangle in which adjacent sides are not equal: The diagonals of a rectangle are perpendicular to each other.

15. Draw about six rectangles of different shapes and write statements about properties of the rectangles which appear to be true. See if you can then find reasons why your statements are true or not true.

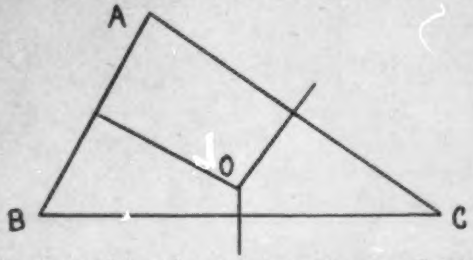
PART G. CONCURRENT LINES

Three or more lines on a point are said to be concurrent lines. The figure shows three concurrent lines on the point K.



A number of sets of concurrent lines are associated with a triangle. For example, the perpendicular bisectors of the sides of a triangle are concurrent. Let us consider why this is true.

In the figure, O is the point of intersection of the perpendicular bisector of BC and the perpendicular bisector of AC.



Since O is on the perpendicular bisector of BC, OB $\overset{m}{=}$ OC.

Since O is on the perpendicular bisector of AC, OA $\overset{m}{=}$ OC.

We conclude that OB $\overset{m}{=}$ OA. We have shown that if OB $\overset{m}{=}$ OA, then O is on the perpendicular bisector of AB. Therefore the perpendicular bisector of AB lies on point O which is the intersection of the other two perpendicular bisectors. The three perpendicular bisectors are concurrent.

EXERCISES

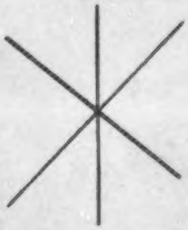
1. Draw the following:

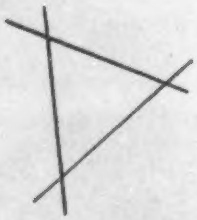
   (a) 3 concurrent lines
   (b) 4 concurrent lines
   (c) 5 concurrent lines

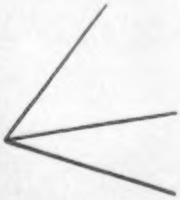
2. Draw three concurrent rays such that the endpoint of the rays is the only element in the intersection set of the three rays.

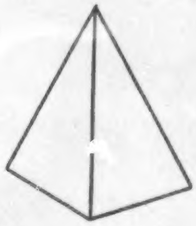
3. How many angles are formed by the rays in exercise 2?

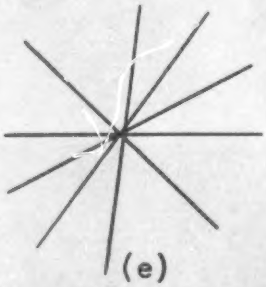
4. Which of the following appear to be concurrent lines or rays? In each figure consider all lines shown.


(a)


(b)


(c)


(d)


(e)

Use a ruler and protractor to draw a triangle with sides lengths 4 inches and 5 inches, and the included angles of measure 60°. Draw the perpendicular bisectors of the 3 sides of the triangle. Do the perpendicular bisectors appear to be concurrent?

6. Repeat exercise 5 with three triangles of different sizes and shapes. You might use a right triangle, an equilateral triangle, and a triangle with one angle greater than 90 degrees in measure. In each, construct the perpendicular bisector of each side.

7. Use ruler and compass to construct other triangles as in exercises 5 and 6 and the perpendicular bisectors of their sides.

8. In a triangle RST, the side ST is said to be opposite vertex R. If the perpendicular line from R to ST intersects ST or ST extended in point V, the segment RV is called the *altitude* of the triangle RST from the point R.

   Use ruler and protractor to draw a triangle of sides of length 3 inches and 4 inches, and included angle of measure 60 degrees. Draw the three altitudes of the triangle.

9. What did you discover about the altitudes in the triangle which you drew in exercise 8? Draw two other triangles and the altitudes of the triangles to see if what seemed to be true in exercise 8 holds for your other triangles

PART H. COMPARISON OF SQUARES

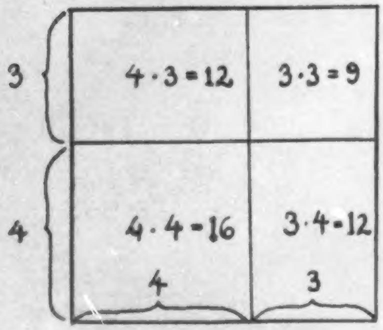


Figure 1

Figure 1 represents a square each side of which measures 7 units. The square has been divided into 4 parts which include:

(a) a square with area measure (4) (4) = 16

(b) a square with area measure (3) (3) = 9

(c) Two rectangles with area measure (4) (3) = 12.

Let us consider a second square equal in area measure but divided into parts in a different way. (If you like, you can think of only one square and division of it in two ways.) Figure 2 represents this second square. This square has been drawn so that

(i) the measures of AF, BG, CH and DE are 4 units.

(ii) the measures of FB, GC, HD, and EA are 3 units

(iii) the four triangles are congruent right triangles, since each of them has two sides which measures 4 units and 3 units, and the included angle for each is a right angle, or an angle of measure 90 degrees.

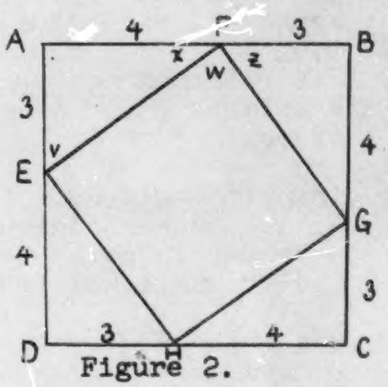


Figure 2.

EXERCISES.

1. Draw a figure like Figure 1 using inches for the units of length in your drawing.
    (a) What is the area measure of the square you have drawn?
    (b) What are the area measures of the two smaller squares in the large square that you have drawn?
    (c) What are the area measures of the rectangles you have drawn?
2. Draw a figure like Figure 2 using inches for the units of length in your figure. Label the points in your figure like those in Figure 2.
3. Give the steps in the argument to show that triangles AEF and BFG in figure 2 are congruent.
4. If the statement (iii) about figure 2 above is true, then EF $\overset{m}{=}$ FG $\overset{m}{=}$ GH $\overset{m}{=}$ HE. Why?

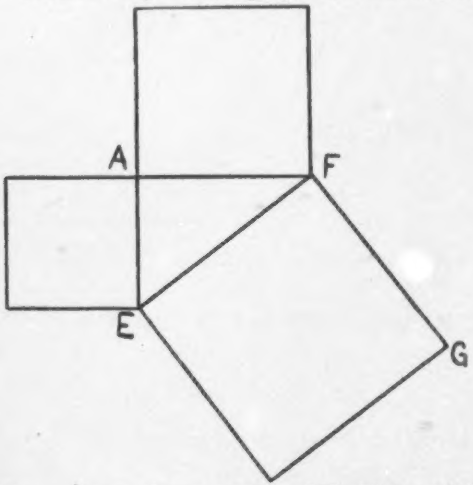
5. Copy the following and complete each statement:
In figure 2 (or exercise 2), $\angle v \overset{m}{=} \angle z$ since triangles ____________ and ____________ are congruent. The sum of the measures of $\angle v$, $\angle x$, and $\angle EAF$ is ________ degrees. The sum of the measures of $\angle v$ and $\angle x$ is ________ degrees. The sum of the measures of $\angle z$ and $\angle x$, and $\angle w$ is ________ degrees. Since $\angle v \overset{m}{=} \angle z$, the sum of the measures of $\angle z$ and $\angle x$ is ________ degrees. So the measure of $\angle w$ is ________ degrees.

6. Using results of exercises 4 and 5, what kind of a quadrilateral is EFGH?(What are the measures of angles FGH, GHE, and HEF?)

7. Cut the square you drew in exercise 1 out of the paper. Then cut each of the two rectangles into two equal triangles. Can you fit these triangles on top of the triangles drawn in exercise 2? Why are these congruent triangles?

8. We have seen in exercise 7 that the sum of the measures of the areas of the two rectangles in figure 1 or exercise 1 are equal to the sum of the measures of the area of the four right triangles in figure 2 or exercise 2. What can you conclude about the sum of the measures of the areas of the two squares in figure 1 (exercise 1) and the measure of the area of the quadrilateral EFGH in figure 2 (exercise 2)? Remember that the two large squares are congruent and the measure of each of their areas is 49.

9. Place the smallest square that you cut out in exercise 7 along AE (your exercise 2) and the larger square along AF as shown in the figure. Make a statement about the squares of the lengths of the two shorter sides of the triangle AFE and the square of the length of the side EF, using your conclusion in exercise 8.



10. Would the same reasoning work if the sides of the triangle AEF had some other lengths? Suppose AE ^{m} 5 inches and AF ^{m} 12 inches. What is the length of EF?

PART I. A RIGHT TRIANGLE PRINCIPLE.

In a right triangle the side opposite the right angle is called the hypotenuse. Since the reasoning in the above exercises applies to any right triangle (see exercise 10), we can state the following general principle:

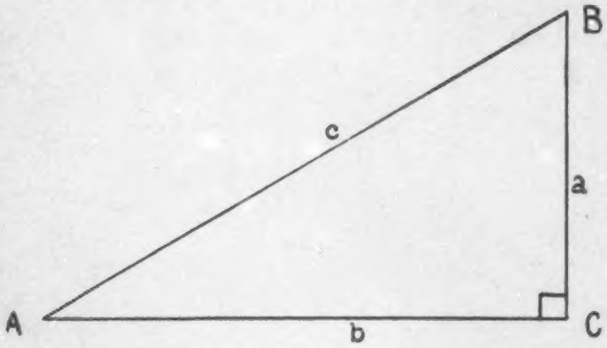
Principle 7: In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

For example, in the above exercises the lengths of AE and AF are 3 and 4 inches respectively. If c is the length of EF, then

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25$$

Therefore c must be a number which, when multiplied by itself, gives 25, so that c must equal 5. Measure EF and check this prediction.

In the right triangle below, c is the length of the hypotenuse and a and b are the lengths of the other two sides.



In each line of the following table two of the three sides of a right triangle are given, and you are to find the length of the third side.

In each case, consider c the hypotenuse.

	a	b	c
a.		12	13
b.		24	25
c.		15	17
d.		21	29
e.		35	37
f.	9		41
g.	28	45	
h.	11		61
i.		33	65
j.	16	63	
k.		55	73
l.		84	85
m.	36	77	
n.	39		89
o.		72	97
p.		99	101

You may find a table of squares useful in solving these problems. When you have filled in the empty spaces in this table, you will have a list of right triangles whose sides are natural numbers.

Such a list has been found on a Babylonian cuneiform tablet dating from between 1600 and 1900 B.C., and now in the Columbia University Library. This shows that the ancient Babylonians must have known Principle 7. Since no practical applications were given, it indicates that almost 4000 years ago some people were working on mathematics for the sheer fun of it. There is no evidence that the Babylonians had any idea of what is meant by a logical proof, and it is believed that they may have found the method for constructing this list by observation and experiment.

Much later the Greek mathematician Pythagoras (when and where did he live?) discovered a proof of this principle. Plutarch (what is he famous for?) tells the story that Pythagoras was so thrilled by the discovery that "he offered a splendid sacrifice of oxen".

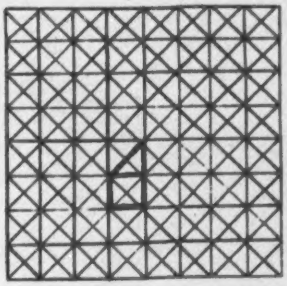
The converse of this principle is also true:

Principle 8: If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

PART J. EXERCISES ON THE PYTHAGOREAN THEOREM.

1. It has been said that Pythagoras noticed the right triangle property by looking at a mosaic like the following:

   Copy the mosaic design and mark with colored pencil a triangle which shows the right triangle property. As a guide a triangle and the square on one side of it have been marked in the figure.



2. As an argument to support the right triangle principle, exercise 1 takes care only of a special case. What is the special case?

3. Show for the following numbers that the square of the first is the sum of the squares of the others in each set of 3:
   (a) 5, 4, 3 (b) 13, 5, 12
   (c) 25, 7, 24 (d) 20, 16, 12

4. Draw or construct triangles with the sides of lengths given in the parts of (a) and (b) of exercise 3. Use your protractor to show that these triangles are right triangles.

5. Draw right triangles the lengths of whose shorter sides are:
   (a) 1 and 2 (b) 4 and 5 (c) 2 and 3.

Measure with a ruler to the nearest one-tenth of a centimeter, if possible, the lengths of the hypotenuses of these triangles. (If you do not have a ruler marked in centimeters, measure to the nearest 1/8 inch.)

6. Use the right triangle principle to find the squares of the lengths of the hypotenuses of the triangles in exercise 5 and check whether principle 6 is true in these cases.

7. In using unsigned numbers, we say that square root of 25 is 5 and the square root of 16 is 4. We would say that square root of 5 is a number such that its square is 5. We know that there is no fraction or whole number of which this is true. We can, however, find a number for which this is true. The approximate values of square roots of some of the whole numbers are given in a table of square roots. Use a table of square roots to find approximate values of the square roots of the following:
(a) 5 (b) 41 (c) 13.

8. Compare the results in exercises 5 and 7. Explain, using the property of right triangles why we might expect the result of exercises 5 (a) and 7 (a) to be about the same. Do the same for the parts (b) and (c) of exercises 5 and 7.

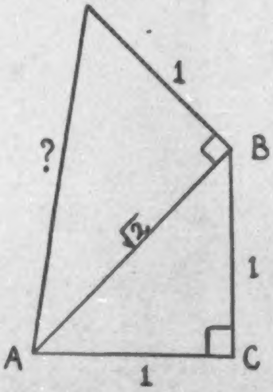
9. Use the property of right triangles to find the lengths of the hypotenuses of right triangles with sides given of the following lengths:
(a) sides of length 3 units and 5 units
The square of the length of the hypotenuse is

$$3^2 + 5^2 = 34.$$

The length of the hypotenuse is the square root of 34. From the table we find that this is 5.8, correct to the nearest one-tenth.
(b) 5 and 6 (c) 10 and 11 (d) 1 and 3.

*10. Draw a square whose sides are of length 1 unit. What is the length of the diagonal? Check by measurement. Now draw a right triangle with the sides 1 unit long. What is the length of the hypotenuse?

*11. Now draw a right triangle of sides "square root of 2" and 1 unit in length as shown in the figure.
In the figure the length of AB is the square root of 2. What is the length of the hypotenuse of this new triangle?



*12. Continue what you have started in exercise 11, and obtain line segments of the following lengths, using your unit:

(a) square root of 3 (b) square root of 4

(c) square root of 5 (d) square root of 6

Your drawing should then look like a spiral curve.

STATEMENT OF PRINCIPLES.

Principle 1: Corresponding segments and corresponding angles of congruent figures have equal measures.

Principle 2: If two angles and the included side of one triangle are equal in measure, respectively, to two angles and the included side of another triangle, then the triangles are congruent.

Principle 3: If the three sides of one triangle are equal in measure, respectively, to the three sides of another triangle, then the triangles are congruent.

Principle 4: If two sides and the included angle of one triangle are equal in measure, respectively, to two sides and the included angle of another triangle, then the triangles are congruent.

Principle 5: If the point P is on the perpendicular bisector of the line segment AB, then the distance from P to the end points A and B are equal.

Principle 6: If the distances from the point P to the points A and B are equal, then P is on the perpendicular bisector of the line segment AB.

Principle 7: In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Principle 8: If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other sides, then the triangle is a right triangle.

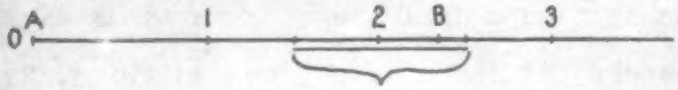
UNIT X

MEASUREMENT AND APPROXIMATION

When you use numbers to count separate objects you need only whole numbers. When you count the number of people in a room you know the result will be a whole number; there will be exactly 11, not 11 1/4 or 10 1/2. If there are a great many people, or you are not sure you have counted correctly, you may say there are "about 300," rounding the the number to the nearest hundred.

When you measure something the situation is different. When you have measured the length of a line segment with a ruler divided into quarter-inches, the end of the segment probably fell between two quarter-inch marks, and you had to judge to which mark it was closer. Although the end seemed to fall almost exactly on a quarter-inch mark, if you had looked at it through a magnifying glass you would probably have found that there was a difference. And if you had then changed to a ruler with the inches divided into sixteenths, you might have decided that the end of the segment was nearer to one of the sixteenth-inch marks than to the quarter-inch mark. Furthermore, as you know, between two points on a line there is always a third point. So scientists and mathematicians agree that measurement of this sort cannot be considered exact, but only approximate; and that the important thing is to know just how inexact a measurement is, and to state it so that it will be clear from the way it is stated how exact it is.

Look at the line below, which shows a scale divided into one-inch units. The zero point is labeled "A", and point B is between the 2-inch mark and the 3-inch mark. Since B is clearly closer to the two-inch mark, we say that segment AB is 2 inches. The fact that we state the measurement "2 inches" implies that AB was measured to the nearest inch. However, any point which is more than 1 1/2 inches from A and less than 2 1/2 inches from A would be the endpoint of a segment which, to the nearest inch, is also 2 inches. The mark below the line shows the space within which the endpoint of a line segment 2 inches long (to the nearest inch) might fall. Such a segment might actually be almost 1/2 inch less than 2, or almost 1/2 inch more than 2 inches. We therefore say that, when a line segment is measured to the nearest whole inch the "greatest possible error" is 1/2 inch. This does not mean that you have made a mistake (or that you have not). It simply means that, if you measure correctly to the nearest whole inch, any measurement more than 1 1/2 inches and less than 2 1/2 inches will be reported in the same way, as 2 inches. Consequently such measurements are sometimes stated as 2 ± 1/2. (The symbol "±" is read "plus or minus.")



Exercises

1. Draw a line and mark on it a scale with divisions of 1/4 inch. Mark the zero point C. Place a point between 1 2/4 and 1 3/4, but closer to 1 3/4, and call the point D. How long is CD to the nearest 1/4 inch?

2. Between what two points on the scale must D lie if the measurement, to the nearest 1/4 inch, is to be 1 3/4? How far from 1 3/4 is each of these points?

3. When you measure to the nearest 1/4 inch, what is the "greatest possible error?"

4. Why may this measurement of CD be stated "1 3/4 ± 1/8?"

5. a. The measurement of a line segment was stated to be 1 1/8 inches. This segment must have been measured to the nearest _______ of an inch.

   b. The endpoint of the segment must have fallen between 1_____ and 1 _____.

   c. The "greatest possible error" in the measurement of this segment is _____.

   d. The measurement might be stated as 1 1/8 ± _____.

6. The measurement of a line segment was stated as 2 5/16 ± 1/32. Between what marks on the scale must the end of this segment lie?

7. When a line segment is measured to the nearest inch, we say the unit used is 1 inch. When it is measured to the nearest 1/2 inch, we say the unit is 1/2 inch. If a measurement is stated to be 5/16 inch, this means the measurement was made to the nearest _____ of an inch, so the unit is _____ inch.
8. The "greatest possible error" in a measurement is always what fractional part of the unit used?
9. Often an inch scale is divided into tenths of an inch. A line segment was measured with such a scale, and stated to be 3 7/10 inches. What was the unit of measurement? What was the greatest possible error? State the measurement 3 7/10 ± _____.
10. Machinists sometimes measure to the nearest 1/100 of an inch. What is the greatest possible error in such a measurement?

Precision

Consider the two measurements, 10 1/8 inches and 12 1/2 inches. What unit was used for each measurement? What is the greatest possible error of each measurement? Since the unit for the first measurement is 1/8 inch, and the unit for the second measurement is 1/2 inch, we say that the first measurement is more precise than the second, or has greater *precision*. Notice also that the greatest possible error of the first measurement is 1/2 of 1/8 or 1/16 inch, and for the second measurement is 1/2 of 1/2 inch, or 1/4 inch. For the first measurement the greatest possible error is less than for the second measurement. So the more precise of two measurements is the one made with the smaller unit, and for which the greatest possible error is therefore the smaller.

It is very important that measurements be stated so as to show correctly how precise they are. If you have measured a line segment to the nearest 1/8 inch, and the measurement is 2 6/8 inches you should not change the fraction to 3/4, for that would make it appear that the unit was 1/4 inch, rather than 1/8 inch. If you measured to the nearest 1/4 inch, and the measurement was closer to 3 inches than to 2 3/4 or 3 1/4 inches you should state it to be 3 0/4, so that it will be clear that the unit used was 1/4 inch.

Exercises

1. Suppose you measured a line to the nearest hundredth of an inch. Which of these numbers would state the measurement best?

   3.2 inches   3.20 inches   3.200 inches

2. Suppose you measured to the nearest tenth of an inch. Which of these numbers should you use to state the result?

   4 inches   4.0 inches   4.00 inches

3. For each of the measurements below tell the unit of measurement, and the greatest possible error. Then tell which measurement in each pair has the greater precision.

   a. 5.2 feet, 2 1/4 feet

   b. .68 feet, 23.5 feet

   c. .235 inches, .146 inches

4. What is your age to the nearest year, that is, what is your nearest birthday - tenth, eleventh, twelfth,...? All of you who say "13" must be between ____ and ____ years old.

5. What is your age to the nearest 1/2 year? All of you who say "12" must have birthdays between the months of ____ and ____.

Usually scientific measurements are expressed in decimal form. For instance, it is known that one meter (a unit in the metric system of measures) is about 39.37 inches. This means that a meter is closer to 39.37 inches than it is to 39.38 inches or 39.36 inches. In other words, one meter lies between 39.375 inches and 39.365 inches.

In the case above one can tell from the way the number is written how precise it is supposed to be. But if we were told that something is 37800 feet long, it is not clear whether the zeros at the end actually indicate the precision. The unit of measurement may have been 100 feet, 10 feet, or 1 foot. Various interpretations might be: (1) the length is closer to 37800 than to 37900 or 37700; (2) the length is closer to 37800 than 37810 or 37790; or (3) the length is closer to 37800 than 37801 or 37799. In a case like this, we frequently underline a zero to show how precise the measurement is. For example, 378$\underline{0}$0 means that the measurement is precise to the nearest 10 feet, while 3780$\underline{0}$ means it was made to the nearest foot. If neither zero is underlined, we understand that the measurement was made to the nearest 100 feet. If a measurement is stated as 5.640 feet we understand, without underlining the zero, that the unit is one-thousandth of a foot, for otherwise the zero would not be written at all.

6. For each measurement below tell what unit of measurement was used and the greatest possible error.

    a. 52700 ft. b. 527$\underline{0}$ ft. c. 527$\underline{0}$0 ft.
    d. 52.7 ft. e. .5270 ft. f. 527.0 ft.

7. Which of the measurements in Ex. 6 is most precise? Which is least precise? Do any two measurements have the same precision?

8. Show by underlining a zero the precision of the following measurements:

    a. 4200 ft., measured to the nearest foot.

b. 23000 miles, measured to the nearest hundred miles.

c. 48,000,000 people, reported to the nearest ten-thousand.

Per Cent of Error

While two measurements may be made with the same precision (that is, with the same unit) and their greatest possible error is therefore the same, this error is more important in some cases than in others. An error of 1/2 inch in measuring your height would not be very misleading, but an error of 1/2 inch in measuring your nose would be misleading. We can get a measure of the importance of the possible error by comparing it with the measurement. Consider these measurements and their possible errors: 4 in. ± .5 in.

58 in. ± .5 in.

Since these measurements are both made to the nearest inch, the greatest possible error in each case is .5 inch. But .5 is a much larger fraction of 4 than it is of 58. If we divide the possible error by each of the measurements we get these results:

$$\frac{.5}{4} = \frac{5}{40} = .125, \text{ or } 12.5\%$$

$$\frac{.5}{58} = \frac{5}{580} = .0086, \text{ or } .86\%$$

We see that in the first case the possible error is 12.5% of the measurement, and in the second the possible error is less than 1% of the measurement.

The percent which the greatest possible error is of the measurement, is called the per cent of error of the measurement.

Exercises

1. State the greatest possible error for each of these measurements.

   a. 52 ft.   b. 4.1 in.   c. 2580 mi.   d. $36\underline{0}$ ft.
   e. 7.03 in.   f. .006 ft.   g. 54,000 mi.   h. $54{,}0\underline{0}0$ mi.

2. Find the per cent of error of each measurement in Exercise 1. (Express your per cent so as to include two non-zero digits.)

3. Find the greatest possible error and the per cent of error for each of the following measurements.

   a. 9.3 ft.   b. .093 ft.   c. 930 ft.   d. 93,000 ft.

4. What do you observe about your answers for Exercise 3? Can you explain why the per cents of error should be the same for all of these measurements?

Significant Digits (Accuracy)

Look again at Exercise 3 above. Notice that in Part a, 9.3 feet, the unit of measurement is .1 foot, so the measurement might be written as follows: 9.3 ft. = 93 x .1 ft.

In 3 b, .093 ft. = 93 x .001 ft.
In 3 c, 930 ft. = 93 x 10 ft.
In 3 d, 93,000 ft. = 93 x 1000 ft.

The units of measure are different, but in each case the number of units is 93. We call the 9 and 3 in 93 significant digits. In measurements, the figures in the numerals which indicate the number of units in the measurement are significant digits.

Exercises

1. Write each measurement below to show the unit of measure and the number of units. Then write the significant digits.

   a. 520 ft.   b. 32.46 in.   c. .002 in.   d. 403.6 ft.
   e. 25,800 ft.   f. .0015 in.   g. 38.90 ft.   h. .0603 in.

2. Find the per cent of error for the following measurements.

   a. 26.3 ft.   b. .263 ft.   c. 2630 ft.
   d. 51,000 mi.   e. 5.1 ft.   f. .051 in.

3. What is the precision of each measurement in Exercise 1 and 2?

The per cent of error of a measurement shows its accuracy. The smaller the per cent of error, the greater is the accuracy of the measurement.

4. Which of the measurements in Exercise 2 have the same accuracy? Which have the same significant digits?

5. How many significant digits are there in each of these measurements?

   a. 52.1 in.   b. 52.10 in.   c. 3.68 in.   d. 368.0 in.

6. Find the per cent of error of each of the measurements in Exercise 5.
7. From your answers for Exercise 5 and 6, can you see any relation between the number of significant digits in a measurement and its per cent of error?
8. Without computing, can you tell which of the measurements below has the greatest accuracy? Which is the least accurate?

23.6 in.   .043 in.   7812 in.   .2 in.

Adding and Subtracting Measurements

Since measurements are never exact, the answers to any questions which depend on those measurements are also approximate. For instance, suppose you measured the length of a room by making two marks along a wall, call them A and B, then measured the distance from the corner to A, from A to B, and then from B to the other corner. Measurements which are to be added should all be made with the same precision. Suppose, to the nearest fourth of an inch, the measurements were 72 1/4 inches, 40 2/4 inches, 22 3/4 inches. You would add these amounts to get 135 2/4 inches. Of course, the distances might have been shorter in each case. They could have been almost as small as 72 1/8, 40 3/8, and 22 5/8 in which case the distance would have been almost as small as 135 1/8 inches, which is three-eighths of an inch less than 135 2/4. Also, each distance might have been longer by nearly one-eighth of an inch, in which case the total length might have been almost three-eighths of an inch longer than 135 2/4. The greatest possible error of a sum is the sum of the possible errors, but usually the errors will to a certain extent cancel each other, one measurement being too long and the other too short.

As we have seen, the greatest possible error of a sum (or difference) of several measurements is the sum of the greatest possible errors of the measurements involved. (To estimate the expected error of a sum, taking into account the way the errors would often cancel each other, we need to use the Theory of Chance. Some ideas involving Chance are given in Unit XIII.)

100

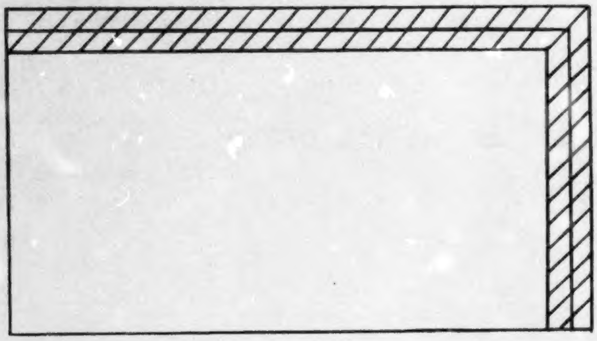
Exercises

Find the greatest possible error of the sums of these measures:

1. 5 1/2 in. + 6 1/2 in. + 3 0/2 in.
2. 3 1/4 in. + 6 1/2 in. + 3 in.
3. 4.2 in. + 5.03 in.
4. 42.5 in. + 36.0 in. + 49.8 in.
5. .004 in. + 2.1 in. + 6.135 in.
6. 2 3/4 in. + 1 5/16 in. + 3 3/8 in.

Approximate Measurement of Area

You know that the area of a rectangle is found by multiplying the number of units in the length by the number of the same units in the width. Suppose that the dimensions of a rectangle are 3 1/4 inches and 1 3/4 inches. Since the measuring was done to the nearest 1/4 of an inch, the measurements can be stated as 3 1/4 ± 1/8 and 1 3/4 ± 1/8. This means that the length might be almost as small as 3 1/8 inches and the width almost as small as 1 5/8 inches. Or the length might be almost 3 3/8 inches and the width almost 1 7/8 inches. Look at the drawing to see what this means. The outside lines show how the rectangle would look if the dimensions were as large as possible. The inner lines show how it would look if the length and width were as small as possible. And the shaded part shows the difference between the largest possible area and the smallest possible area with the given measurements.



To figure out these areas, we multiply 3 1/8 x 1 5/8 to find the smallest possible area, and 3 3/8 x 1 7/8 to find the area if it is as large as possible.

$$3\ 1/8 \times 1\ 5/8 = \frac{25}{8} \times \frac{13}{8} = \frac{325}{64} = 5\ 5/64$$

$$3\ 3/8 \times 1\ 7/8 = \frac{27}{8} \times \frac{15}{8} = \frac{405}{64} = 6\ 21/64$$

These results show that there seems to be a difference of more than 1 square inch in the two possible areas.

If we find the area by using the measured length and width, we find that

$$3\ 1/4 \times 1\ 3/4 = \frac{13}{4} \times \frac{7}{4} = \frac{91}{16} = 5\ \frac{11}{16}$$

However, since we have seen that the area might be either larger or smaller than this number of square inches, it would not be correct to give the result in this way, which means that the area has been found to the nearest 16th of a square inch. The area could be as much as 41/64 square inches greater or 39/64 square inches less than 5 11/16. Since both 41/64 and 39/64 are about 40/64, which is 5/8, we could express the area as 5 11/16 ± 5/8 square inches. One good way of expressing the result to indicate the accuracy would be to write 5 1/2 square inches since the area cannot be less than 5 and not much more than 6.

Exercise

1. Suppose a rectangle is 2 1/2 inches long and 1 1/2 inches wide. Make a drawing of the rectangle. Show on the drawing that the length is 2 1/2 ± 1/4 and the width 1 1/2 ± 1/4. Then find the largest area possible and the smallest area possible, and find the difference, or uncertain part. Then find the area with the measured dimensions, and find the result to the nearest 1/2 square inch.

We can see a little better what is happening in general if we do the problem above somewhat differently. We noticed that the dimensions of the rectangle might be as big as (3 1/4 + 1/8) and (1 3/4 + 1/8). Suppose we multiply these numbers without combining. (We are using the distributive property). Then we would have

$$\begin{aligned}(3\ 1/4+1/8)(1\ 3/4+1/8) &= (3\ 1/4+1/8)(1\ 3/4)+(3\ 1/4+1/8)(1/8)\\ &= (3\ 1/4\times 1\ 3/4)+(1/8\times 1\ 3/4)+(3\ 1/4\times 1/8)+(1/8\times 1/8)\\ &= (3\ 1/4\times 1\ 3/4)+(1/8\times 1\ 3/4)+(1/8\times 3\ 1/4)+(1/8\times 1/8)\\ &= (3\ 1/4\times 1\ 3/4)+(1/8)(1\ 3/4+3\ 1/4)+(1/8\times 1/8)\end{aligned}$$

In the last line above, the first product is just the product of the measurements. The second product is the amount of possible error multiplied by the sum of the approximate measurements. The last product is very small. Hence the error in the number of square units of the computed area is just about equal to the product of the possible error in the linear units multiplied by the sum of the approximate lengths. To check this against our previous computation notice that $(1/8)\ (1\ 3/4 + 3\ 1/4) = (1/8)\ (20/4) = 5/8$.

We noticed in the example which we just finished that the error in the number of square units of the computed area is just about the same as the product of the greatest possible error in the linear units (assuming the linear measurements are of the same precision) multiplied by the sum of the approximate lengths. One can see that this would be true for any numbers by seeing how the above example works. A slightly better way of seeing this is to use letters in place of the numbers. We could use the letter a in place of 3 1/4, and the letter b in place of 1 3/4. Then the measured length of the rectangle would be a inches and the width would be b inches. The computed area would be the product of a and b which is usually written ab. Then if the possible error is 1/8, the dimensions could be as large as $(a + 1/8)$ and $(b + 1/8)$. Then the area could be as large as

$$(a + 1/8)\ (b + 1/8) = ab + (1/8)b + a(1/8) + (1/8)(1/8).$$

Since a and b are numbers, $a(1/8)$ is equal to $(1/8)a$. Then, using the distributive property, we have

$$(a + 1/8)(b + 1/8) = ab + (1/8)(a + b) + (1/8)(1/8).$$

The product ab is just the product of the approximate measurements. The next product $(1/8)(a + b)$ is the amount of possible error in the linear measurements multiplied by the sum of the approximate measurements. The product $(1/8)(1/8)$ is very small. Thus we have the same general conclusion that we had before. Furthermore, we could replace a and b by any other measurements and the general result would be the same, provided that a and b are equally precise.

Exercises

1. If instead of considering how large the dimensions of the rectangle in the above example could be, we considered how small they could be, that is $(3\ 1/4 - 1/8)$ and $(1\ 3/4 - 1/8)$, find the product as we did above and draw similar conclusions.
2. Suppose the lengths of the sides of a rectangle are measured to be 46.2 and 23.4 inches precise to the nearest tenth; that is, that the greatest possible error is .05. About how large is the possible error in the area found by multiplying the measurements of the sides?

3. Suppose the lengths of the sides of a rectangle are measured to the nearest tenth of an inch and that each length is less than 100 inches. By how much may the true value of the area differ from the computed value?
4. If c and d represent two measurements given with a greatest possible error of .1, what is the possible error of the product of c and d?
5. Answer the previous question if the greatest possible error is .01 instead of .1.